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Co-growth in nilpotent groups

Wilson, Aaron Thomas

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Aaron Thomas Wilson

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Signature of Author ...A T Wilson...

Aaron Thomas Wilson

To everyone who helped

"We are all in the gutter, but some of us are looking at the stars."
– Oscar Wilde

"I can't even think straight."
– Anon

Abstract

The problem of subgroup growth (how may subgroups of a given index does a group have) is well-known, and for finitely genetrated torsion free nilpotent groups, we can calculate zeta functions by means of good bases [GSS88]. We study the converse problem, which we call co-growth, i.e. how many overgroups of a given co-index does a finitely genetrated torsion free nilpotent group have. As with subgroup growth we calculate zeta functions for these overgroups by adapting the good basis method to this new situation. By applying the Tauberian Theorem we gather information on the asymptotics of the co-growth of these overgroups.

We also, as with the subgroup case consider not just overgroups of finitely generated torsion free nilpotent groups but also overgroups which are isomorphic to the original group and those overgroups in which the orignal group is normal.

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Chapter 1

Introduction

We begin by considering some mathematical background.

1.1 Dirichlet Series

A Dirichlet Series is an expression of the following form

$$f(s) = \sum_{n \in \mathbb{N}} a_n n^{-s}$$

where each $a_n \in \mathbb{C}$. Our main use of Dirichlet series will be as a formal sum, where the quantity $a_i \in \mathbb{Z}$ counts some aspect of the structure of a group G. However, instead of regarding a Dirichlet series as a formal gadget, such a series may be viewed analytically as a function of s, with the domain of s being the subset of the complex plane in which the series converges. Sometimes we will address the issue of convergence and we note that for any Dirichlet series $f(s)$, exactly one of the following will occur:

a. $f(s)$ will converge for all s

b. $f(s)$ will converge for no s

c. there is a real number r such that $f(s)$ converges if $\mathrm{Re}(s) > r$ and is not convergent if $\mathrm{Re}(s) < r$.

We now define the abscissa of convergence of a Dirichlet series $f(s)$ - this is the infimum of values for which the series converges. That is ∞ in case (a) above, $-\infty$ in case (b) and r in case (c) above. We can apply the Tauberian Theorem (see [Lan64, Chapters VII and XV]) to gain group theoretic information from the Dirichlet series we calculate. The Tauberian theorem provides us with asymptotic estimates for the partial sum of the first m coefficients of a Dirichlet series.

Theorem (Tauberian Theorem, see [Lan64, Theorem XV.3.1]). *Let*

$$f(s) = \sum_{n\in\mathbb{N}} a_n n^{-s}$$

be a Dirichlet series with non-negative real coefficients. Define

$$s_r = \sum_{n<r} a_n$$

then if $f(s)$ may be extended analytically to the line $\mathrm{Re}(s) = \alpha_f$ *except for a simple pole at α_f we then have*

$$s_r \sim \frac{Ar^{\alpha_f}}{\alpha_f}.$$

for some $A \in \mathbb{R}$. (Note that by saying $f(s)$ may be extended analytically to the line $\mathrm{Re}(s) = \alpha_f$ *except for a simple pole at α_f we mean that we may write*

$$f(s) = \frac{g(s)}{s-\alpha_f} + h(s)$$

where $g(s)$ and $h(s)$ are holomorphic on the closed right half-plane in question).

This then gives the best possible upper bound for the degree of growth of s_r. We note that this statement may be generalised to the case where there exists a double pole at α_f. In this case we can say (if there is an appropriate analytic completion), that there exists a $B \in \mathbb{R}$ such that

$$s_r \sim Br^{\alpha_f} \log r \quad \forall r \leq c > 1$$

We now define the formal sum and product of two Dirichlet series,

Definition. *Let*

$$f(s) = \sum_{n\in\mathbb{N}} a_n n^{-s} \text{ and } g(s) = \sum_{n\in\mathbb{N}} b_n n^{-s}$$

then we define

$$\begin{aligned} f(s) + g(s) &= \sum_{n \in \mathbb{N}} (a_n + b_n) n^{-s} \\ f(s) \cdot g(s) &= \sum_{n \in \mathbb{N}} \left(\sum_{d|n} a_{n/d} b_d \right) n^{-s}. \end{aligned}$$

The best known Dirichlet series is the classical Riemann Zeta Function

$$\zeta(s) = \sum_{n \in \mathbb{N}} n^{-s}.$$

Although named after Riemann, who wrote a fundamental paper on its properties, this function was in fact introduced by Euler. Euler showed that this sum may be written as a product

$$\zeta(s) = \prod_{p \in \mathcal{P}} \zeta^p(s)$$

where $\mathcal{P}$ denotes the set of rational primes and we set

$$\zeta^p(s) = \frac{1}{(1 - p^{-s})} = \sum_{n=0}^{\infty} p^{-ns}.$$

This is a very powerful result since it allows methods of analysis to be applied to the study of prime numbers. It is essentially the Fundamental Theorem of Arithmetic. Riemann's great advance was to view s as a complex variable, instead of a real variable as Euler had done. This then allowed complex function theory to be applied to zeta functions. One of the great unsolved problems of mathematics involves the zeros of the Riemann Zeta Function, and a solution of this would resolve many questions involving the distribution of prime numbers.

We now turn aside from Dirichlet series and look at some group theory, before uniting the two together later.

1.2 Group Theory

Let G be a group. Let $g, h \in G$ be any elements of G then the commutator of g and h is

$$[g, h] = g^{-1}h^{-1}gh$$

and that we are using left normed commutators so we put

$$[x_1, x_2, x_3] = [[x_1, x_2], x_3].$$

The following formula's of P. Hall are useful when dealing with commutators:

$$\begin{aligned} [x, yz] &= [x, z][x, y]^z \\ [xy, z] &= [x, z]^y[y, z] \end{aligned}$$

see [Hal79, Lemma 2.1].

If $H, K \leq G$ we then let $[H, K] = \langle [h, k] | h \in H, k \in K \rangle$. We now define the fully invariant groups $\gamma_i(G)$ by setting $\gamma_1(G) = G$ and then inductively putting

$$\gamma_{i+1}(G) = [\gamma_i(G), G].$$

These groups $\gamma_i(G)$ then form the lower central series of G and we have

$$G = \gamma_1(G) \trianglerighteq \gamma_2(G) \trianglerighteq \cdots .$$

Note that $\gamma_i(G) \trianglelefteq G$ for every i by induction on i. We define another normal series of a group by letting $Z_0(G) = 1$ and inductively defining

$$Z_i(G) = \{g \in G | [g, h] \in Z_{i-1}(G) \quad \forall h \in G\}.$$

Note that this is a characteristic series and that $Z_1(G)$ is the centre of G. The upper central series of a group G is then

$$1 = Z_0(G) \trianglelefteq Z_1(G) \trianglelefteq \cdots .$$

More generally a finite series of subgroups

$$1 = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_r = G$$

of a group G with $[G_i, G] \leq G_{i-1}$ is called a central series. A group with such a central series is called nilpotent and the minimum length of such a series is called the nilpotency class of G. In this case both the lower and upper central series achieve this lower bound so that $\gamma_{c+1}(G) = 1$ and $Z_c(G) = G$. Simple examples of nilpotent groups are the trivial group (which has class 0) and non-trivial abelian groups (which have class 1).

We will mainly be concerned with infinite groups. However, as it will be useful, we will give the following characterisation of finite nilpotent groups.

Theorem (see for example [Rob96, Theorem 5.2.4]). *Let G be a finite group. Then the following are equivalent.*

i. The group G is nilpotent.

ii. Every subgroup of G is subnormal.

iii. The group G satisfies the normaliser condition (i.e. if $H \lneqq G$ then $H \lneqq N_G(H)$).

iv. Every maximal subgroup of G is normal.

v. G is the Cartesian product of its Sylow p-subgroups.

Note that for infinite groups properties (iii)–(v) are weaker then nilpotency. Until recently this was also the case for property (ii), but a partial converse has now been found. It is clear that if G is nilpotent, then every subgroup is subnormal since if $H \leq G$ then $HZ_i(G) \trianglelefteq HZ_{i+1}(G)$ because $Z_i(G)/Z_{i+1}(G) \leq Z(G/G_i)$. Hence we have

$$H = HZ_0(H) \trianglelefteq HZ_1(G) \trianglelefteq \cdots \trianglelefteq HZ_c(G) = G$$

is a subnormal series for H. Thus any subgroup is subnormal in at most c steps. Howard Smith has recently established:

Theorem. *[Smi00, Theorem 1] Let G be a torsion free group with all subgroups subnormal. Then G is nilpotent.*

In this thesis we will mainly be concerned with a particular class of nilpotent groups — the torsion free finitely generated nilpotent groups. We will denote the class of all such groups by $\mathcal{T}$. The free nilpotent groups are of particular interest. We define them as follows:

Definition. *Let F_n denote the free group on n letters. We will write*

$$F_n^c = F_n/\gamma_{c+1}(F_n)$$

for the free nilpotent group *of class c on n letters.*

In fact $F_n^c \in \mathcal{T}$ and in this particular case the upper and lower central series coincide. We also note the following formula of Witt :

$$\text{rank}\,(\gamma_j(F_n^c)/\gamma_{j+1}(F_n^c)) = \frac{1}{j}\sum_{d|j} \mu(d) n^{j/d}$$

where μ is the is the Möbius function, $\mu(1) = 1$, $\mu(d) = (-1)^r$ if d is a product of r distinct primes, otherwise $\mu(d) = 0$. One particular free nilpotent group we will be concerned with is the group F_2^2 – sometimes called the Heisenberg group. This can be described in mays ways, a particularly concrete one being as a group of upper unitriangular matrices,

$$\left\{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \middle| x, y, z \in \mathbb{Z} \right\}.$$

A presentation for this group is

$$F_2^2 = \langle x, y, z | [x, y] = z, [x, z] = [y, z] = 1 \rangle$$

and we note that this group has Hirsch length 3. Following [Smi83, Definition 1.5] we make the following definition.

Definition. *A $\mathcal{T}$-group of nilpotency class* 2 *and Hirsch length* 3 *will be called an $\mathcal{H}$*-group.

It is also known that if $G \in \mathcal{H}$ then G is commensurable with F_2^2 and that then the following lemma allows us to observe that for $G \in \mathcal{H}$ we have,

$$G \simeq \langle x, y, z | [x, y] = z^t, [x, z] = [y, z] = 1 \rangle.$$

Lemma 1.1 ([Smi83, Lemma 1.5]). *If $G \in \mathcal{H}$ then G is classified up to isomorphism by the single numerical invariant $|Z(G) : \gamma_2(G)|$.*

We now recall the following.

Definition. *Let $\mathfrak{P}$ be a property of groups. Then a group G is*

1. poly-$\mathfrak{P}$ *if there is a finite chain of subgroups*

$$G = G_n \trianglerighteq G_{n-1} \trianglerighteq \cdots \trianglerighteq G_1 \trianglerighteq G_0 = 1$$

such that each factor G_i/G_{i-1} has property $\mathfrak{P}$.

2. residually-$\mathfrak{P}$ *if*

$$\bigcap\{N | N \trianglelefteq G \text{ and } G/N \text{ has } \mathfrak{P}\} = 1$$

(in other words if $\forall g \in G$ with $g \neq 1$ there exists $N \trianglelefteq G$ such that G/N is $\mathfrak{P}$ and $g \notin N$).

Thus for example, we call a group G polycyclic if G has a finite series of subgroups

$$1 = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_n$$

such that G_i/G_{i-1} is cyclic for $i = 1, 2, \ldots, n$. The number of infinite cyclic factors in such a series is called the Hirsch length of G – denoted $h(G)$ – (and is independent of the series we choose). If we have the stronger condition that $G_i \trianglelefteq G$ for every i we say that G is supersoluble. We note that polycyclic groups are residually finite (see for instance [Seg83, Theorem 1.C.1] and [Bau71, Corollary 1.21]).

As a finitely generated nilpotent group has a central series in which every factor is cyclic, we have that finitely generated nilpotent groups are supersoluble. It is also clear that supersoluble groups are polycyclic. We also have the result that finitely generated nilpotent groups are supersoluble and polycyclic and thus residually finite. As any subgroup of a polycyclic group is polycyclic, we have that a subgroup of a finitely generated nilpotent group will be finitely generated. We also have the following result of Gruenberg, which will be useful later.

Theorem (Gruenberg, see for example [Rob96, Theorem 5.2.21]). *If $G \in \mathcal{T}$ then G is residually finite-p for all primes $p \in \mathcal{P}$.*

1.3 Mal'cev Completions

In this thesis we will be considering the number of overgroups of a group G. In most cases G will be a $\mathcal{T}$-group and here we count overgroups within the Mal'cev completion $G^{\mathbb{Q}}$ of a $\mathcal{T}$-group. This construction is due to A. I. Mal'cev [Mal49]. We will now briefly give one possible construction of this completion and outline another. Later on we will give the construction which we will be using. For the first construction we follow [Bau71].

Firstly we introduce the notion of isolators.

Definition. *Let π be a set of primes (i.e. $\pi \subseteq \mathcal{P}$).*

1. *Let $n \in \mathbb{N}$. We say n is a* π-number *if whenever $q \mid n$ and $q \in \mathcal{P}$ then $q \in \pi$.*
2. *Let G be a group. Then G is a* π-group *if every element $g \in G$ has order a π-number.*

Given a nilpotent group G and a subgroup $N \leq G$ we define the π-isolator of N in G as follows,

$$I_G^{\pi}(N) = \{g \in G | \exists m \in \mathbb{N}, m \text{ a } \pi\text{-number such that } g^m \in N\}.$$

Although defined as a set, $I_G^{\pi}(N)$ is in fact a group in the event that G is nilpotent. We then say a subgroup $H \leq G$ is π-isolated if we have for $g \in G$ and $p \in \pi$,

$$g^p \in H \implies g \in H \ (i.e.\ I_G^{\pi}(H) = H).$$

If we take $\pi = \mathcal{P}$ we often omit the superscript and write $I_G(N)$ for $I_G^{\pi}(N)$. If $I_G(N) = N$ we say that N is isolated in G. A result due to Kontorvic (see [Bau71]) is that the terms of the upper central series of a $\mathcal{T}$-group are isolated.

We now show the embedding;

Lemma ([Bau71, lemma 2.3]). *Let $G \in \mathcal{T}$ and G be of nilpotency class c with $g \in G$ and $n \in \mathbb{N}$. Then G can be embedded in a torsion free group of class c in which g has an n^{th} root.*

Proof. [Bau71] Pick a prime p such that $p \not| n$. Since G is residually-finite-p (being a $\mathcal{T}$-group) we can find subgroups $N_1, N_2, \ldots$ of G of p-power index such that

$$\bigcap_{i \in \mathbb{N}} N_i = 1.$$

Let P be the Cartesian product of the groups G/N_i, *i.e*

$$P = \prod_{i \in \mathbb{N}} G/N_i.$$

The projection $\varphi : G \to P$ mapping $g \mapsto (gN_i)$ is a monomorphism since the N_i intersect in 1. Here we note that (gN_i) is the element of P which has gN_i as its component in G/N_i. Now, gN_i has an n^{th} root h_i in G/N_i. Hence $h = (h_i)$ is an n^{th} root of $(g)\varphi$. We observe that P is nilpotent of class c since none of the G/N_i has class exceeding c. Thus $H = \langle (G)\varphi, h \rangle$ is of class c. Finally since $(G)\varphi \cap \tau(H) = 1$ ($\tau(H)$ begin the torsion group of H). It follows that G embeds into $H/\tau(H)$ as required.

The fact that we can find an n^{th} is due to the fact that if Q is a p-group and $p \not| n$ then we can find an a and b such that $ap^k + bn = 1$ for some $k \in \mathbb{N}$. Hence, if g has order p^k we have

$$g = g^1 = g^{ap^k + bn} = (g^{p^k})^a g^{bn} = (g^b)^n$$

so that g^b is an n^{th} root of g. This root is unique as if $x \in Q$ with $x^n = g$ we have

$$x = x^1 = x^{ap+bn} = x^{ap} x^{bn} = 1g^b$$

since Q is a p-group.

□

This can be carefully extended so that every $\mathcal{T}$-group can be embedded into a group where roots can always uniquely be extracted. This uses some of the theory of isolators outlined above. As a first step we need to show

Lemma 1.2 ([Bau71, Lemma 2.7]). *Let A and B be $\mathcal{T}$-groups and H and K be torsion free overgroups of A and B respectively. Let $x \in H$ and $y \in K$ be such that*

$$H = \langle A, x \rangle \text{ and } K = \langle B, y \rangle.$$

Let $n \in \mathbb{N}$ be such that

$$x^n = a \in A \text{ and } y^n = b \in B.$$

If φ is a homomorphism of A onto B which maps a onto b then φ can be extended to a homomorphism of H into K in such a way that x is mapped to y.

We have the following immediate consequence of this,

Corollary 1.3 ([Bau71, Corollary 2.71]). *Let A, B, H and K be as above. If φ is an isomorphism then so is its extension to H.*

From this it follows that there is essentially only one way to adjoin an n^{th} root to an element in a $\mathcal{T}$-group if the resulting group is also to be torsion free nilpotent. This then enables one to show

Theorem (Mal'cev). *Let $G \in \mathcal{T}$. Then G can be embedded into a torsion free nilpotent group where extraction of roots is unique and always possible.*

A minimal torsion free group containing a given $\mathcal{T}$-group G as above is a Mal'cev completion of G, which we will denote $G^{\mathbb{Q}}$. An immediate consequence of this approach to the existence of $G^{\mathbb{Q}}$ is the following lemma.

Corollary 1.4 ([Bau71, Corollary 2.41]). *Let $G, H \in \mathcal{T}$ and $\varphi : G \to H$. If $G^{\mathbb{Q}}$ and $H^{\mathbb{Q}}$ are any Mal'cev completions of G and H then φ can be uniquely extended to $\varphi^{\mathbb{Q}} : G^{\mathbb{Q}} \to H^{\mathbb{Q}}$.*

This yields further

Corollary 1.5 ([Bau71, Corollary 2.42]). *Let $G \in \mathcal{T}$. Any two Mal'cev completions of G are isomorphic.*

Thus we may say *the* Mal'cev completion $G^{\mathbb{Q}}$ of a $\mathcal{T}$-group G.

Another more concrete way to build the Mal'cev completion of a $\mathcal{T}$-group is to use a theorem of Auslander, which was first conjectured by P. Hall in the 1950's, and involves embedding G into upper triangular groups over the integers.

Theorem (Auslander see [Hal79] or [Seg83]). *Every $\mathcal{T}$-group is isomorphic with a subgroup of $\mathrm{Tr}_1(n, \mathbb{Z})$, where Tr_1 is the group of upper-triangular matrices with diagonal entries 1 (the group of unitriangular matrices).*

It is then clear that we may embed

$$\mathrm{Tr}_1(n,\mathbb{Z}) \hookrightarrow \mathrm{Tr}_1(n,\mathbb{Q})$$

which is visibly torsion free nilpotent and radicable. We can then define,

$$G^{\mathbb{Q}} = \{g|g \in \mathrm{Tr}_1(n,\mathbb{Z}), g^t \in G \text{ for some } t \in \mathbb{Z}\}$$

Although this approach is concrete, it clouds the universal properties which are clearly visible in the previous approach.

1.4 Subgroup Growth

The main use of Dirichlet series in group theory is in the subject of subgroup growth. The idea of associating a Dirichlet series to a group was motivated by the Dedekind zeta function of a number field. This Dirichlet series is defined for a ring of algebraic integers $\mathfrak{O}$ over a field $\mathfrak{F}$,

$$\zeta_{\mathfrak{F}}(s) = \sum_{n\in\mathbb{N}} r_n n^{-s}$$

where r_n is the number of ideals of $\mathfrak{O}$ of index n and captures some of the algebraic structure of $\mathfrak{F}$. This was the motivation behind Smith [Smi83] and Grunewald, Segal and Smith [GSS88] who developed the zeta function of a finitely generated group which they defined as follows:

Definition. *Let G be a finitely generated group. The* zeta function of G *is*

$$\zeta_G(s) = \sum_{H\leq_f G} |G:H|^{-s} = \sum_{n\in\mathbb{N}} a_n n^{-s}$$

where $a_n = a_n(G) = |\{H|H \leq G, |G:H| = n\}|$. (Note that by $H \leq_f G$ we mean that $H \leq G$ and $|G:H| < \infty$.)

We note that the restriction to finitely generated groups means that $a_n(G) < \infty$ for every $n \in \mathbb{N}$. If this is the case anyway, we may relax the restriction on finite generation. As first example we note that

$$\zeta_{C_\infty^d}(s) = \zeta(s)\zeta(s-1)\cdots\zeta(s-d+1).$$

Although this is a classical result there are four proofs of this given in [Lub93]. We will give one if these later on. Other zeta functions are known and some for non-nilpotent groups. For example McDermott in his thesis [McD97] has calculated the zeta function of the seventeen plane crystallographic groups.

As we have an Euler product for the Riemann zeta function we would like such a factorisation to exist for the group zeta function. This is possible in the case of $\mathcal{T}$-groups and we show this now. Following [Smi83] let $G \in \mathcal{T}$ and $H \leq_f G$ be such that

$$n = |G : H| = \prod_{p \in \mathcal{P}} p^{\lambda_p}$$

then there exists

$$\mathcal{K} = \{R_p | R_p \leq G, |G : R_p| = p^{\lambda_p}, p \in \mathcal{P}\}$$

with $H = \cap_{p \in \mathcal{P}} R_p$. Moreover $\mathcal{K}$ is determined by H. We note that for all but a finite number of primes $p \in \mathcal{P}$, we have $R_p = G$. Conversely given a set $\mathcal{K}$ as above, with the property that $R_p = G$ for all but a finite number of primes then

$$H = \bigcap_{p \in \mathcal{P}} R_p$$

satisfies $H \leq_f G$. Thus we have shown

Lemma 1.6. *Let $G \in \mathcal{T}$ then*

$$\zeta_G(s) = \prod_{p \in \mathcal{P}} \zeta_G^p(s)$$

where

$$\zeta_G^p(s) = \sum_{\substack{H \leq_f G \\ |G:H| \text{ a } p\text{-power}}} |G : H|^{-s}.$$

As $\zeta_G(s)$ is a Dirichlet series it will have a abscissa of convergence and we will write α_G for this (instead of α_{ζ_G}). This information along with the Tauberian theorem allows us to make asymptotic estimates for subgroup growth of groups for which we have zeta functions. An obvious question to now ask is for which groups do we have certain types of subgroup growth. For example we know

Theorem 1.7 (Lubotzky-Mann-Segal (see [Lub93])). *Let G be a finitely generated residually finite group. Then G has polynomial subgroup growth (PSG)* *if G is virtually soluble of finite rank.*

We recall that a group is virtually soluble if it contains a soluble group of finite index. A group is of finite rank if the rank of every finitely generated subgroup is bounded by a given number of generators.

Also recently [Klo00] has given a relationship between α_G and the Hirsch length of a group. Thus we have

Theorem 1.8 ([Klo00]). *Let G be a residually finite virtually soluble minimax group. Then*
$$\frac{h(G)}{7} \leq \alpha_G \leq h(G) + 1$$

A minimax group is a group with a series of finite length whose factors all satisfy max *or* min. As noted in [Klo00] this lower bound is not sharp, see for instance the calculation of $\zeta_H(s)$ where $H = F_2^2$ is the Heisenberg group.

1.5 Co-growth of a group

The notion of subgroup growth of a group has been extensively studied recently. We now introduce the analogous idea of co-growth of a group, which seems not to have been considered. To start with, as with the beginnings of subgroup growth, we consider only $\mathcal{T}$-groups. Thus we make the following definition.

Definition. *The* co-index zeta function *of a torsion free finitely generated nilpotent group G is*
$$\zeta_G^{up}(s) = \sum_{\substack{G \leq_f T \leq G^{\mathbb{Q}} \\ |T:G| < \infty}} |T : G|^{-s} = \sum_{n \in \mathbb{N}} b_n n^{-s}$$
where $G^{\mathbb{Q}}$ is the Mal'cev completion of G and
$$b_n = b_n(G) = |\{T \leq G^{\mathbb{Q}} | G \leq T, |T : G| = n\}|.$$

(One reason to restrict ourselves to $\mathcal{T}$-groups is that $G^{\mathbb{Q}}$ will make sense in this case – in other circumstances $G^{\mathbb{Q}}$ will have to be replaced by a suitable ambient group.) For such a Dirichlet series we will denote $\alpha_{\zeta_G^{\mathrm{up}}}$ – its abscissa of convergence by $\alpha_{\zeta_G^{\mathrm{up}}}$.As with both the Riemann Zeta Function and the zeta function of $\mathcal{T}$-groups it would be helpful to have an Euler product formula for this zeta function. We prove that this zeta function has an Euler product using the next two lemmas 1.9 and 1.10.

Lemma 1.9. *Let $G \in \mathcal{T}$ and for each $p \in \mathcal{P}$ select an overgroup Q_p such that $G \leq Q_p \leq G^{\mathbb{Q}}$ with $|Q_p : G| = p^{\lambda_p}$. Denote this collection $\mathcal{Q} = \{Q_p | p \in \mathcal{P}\}$. If we have that $Q_p = G$ for all but finitely many primes p (i.e. $|\mathcal{Q}| < \infty$) then the group*

$$T = \langle Q | Q \in \mathcal{Q} \rangle$$

is such that

$$G \leq_f T \text{ and } |T : G| = \prod_{p \in \mathcal{P}} p^{\lambda_p}.$$

Proof. Firstly let us rewrite the (finite) collection $\mathcal{Q}$ as

$$\mathcal{Q} = \{Q_i | i = 1, 2, \ldots n\} \cup G$$

with for each i, $|Q_i : G| = p_i^{\lambda_{p_i}}$ for $p_i \in \mathcal{P}$ We note that $Q_i \cap Q_j = G$ if $i \neq j$. Let T be as above and define

$$m = \prod_{p \in \mathcal{P}} p^{\lambda_p}.$$

We form the group $G^{\frac{1}{m}} = \langle g \in G^{\mathbb{Q}} | g^m \in G \rangle$ and observe that $|G^{\frac{1}{m}} : G|$ is finite by the theory of isolators in nilpotent groups (see [Hal79, Chapter 4]). As $|Q_i : G| = p_i^{\lambda_{p_i}}$, if $g \in Q_i$, then $g^{p_i^{\lambda_{p_i}}} \in G$. Hence $Q_i \leq G^{\frac{1}{m}}$ for each $i = 1, 2, \ldots, n$. Therefore we have that $T \leq G^{\frac{1}{m}}$, so that $G \leq_f T$.

It remains to show that $|T : G| = m$. Let $C = \mathrm{Core}_T(G)$ and note that $|T : C| < \infty$ and $C \trianglelefteq T$. (Recall that $\mathrm{Core}_T(G) = \cap_{t \in T} G^t$, and that if T is finitely generated with $|T : G| < \infty$ then $|T : \mathrm{Core}_T(G)| < \infty$.) We can now factor out C and work with finite nilpotent groups. Thus let $\pi : T \to T/C$ be the canonical projection and form the groups,

$$\begin{aligned} \overline{T} &= (T)\pi, \\ \overline{Q}_i &= (Q_i)\pi, \\ \overline{G} &= (G)\pi. \end{aligned}$$

Let $p_1, p_2, \ldots, p_r$ be the primes involved in $|\overline{T}|$. We note that this list includes at least $p_1, p_2, \ldots, p_n$ above, as $\overline{Q}_i \leq \overline{T}$. Now $\overline{T}$ is a finite nilpotent group and hence we can write $\overline{T}$ as the product of its Sylow p-subgroups,

$$\overline{T} = \overline{T}_1 \times \overline{T}_2 \times \cdots \times \overline{T}_r$$

where for each i, $\overline{T}_i$ is the Sylow p_i-subgroup of $\overline{T}$. We do the same for $\overline{G}$,

$$\overline{G} = \overline{G}_1 \times \overline{G}_2 \times \cdots \times \overline{G}_r$$

where for each i we have $\overline{G}_i$ being the Sylow p_i subgroup of $\overline{G}$ (note that some factors may be trivial). Since $\overline{G} \leq \overline{T}$ we have $\overline{G}_i \leq \overline{T}_i$ for every i.

We note that as $\overline{G} \leq \overline{Q}_i$ and $|\overline{Q}_i : \overline{G}| = p_i^{\lambda_i}$ we have that

$$\overline{Q}_i = \overline{G}_1 \times \overline{G}_2 \times \cdots \times \overline{K}_i \times \overline{G}_{i+1} \times \overline{G}_r.$$

It is easy to see that $K_i = T_i$ so that we have,

$$|\overline{T} : \overline{G}| = |T : G| = \prod_{i=1}^{n} p_i^{\lambda_{p_i}}.$$

□

We now prove the converse of this lemma,

Lemma 1.10. *Let T be an overgroup of G with $G \leq_f T \leq G^{\mathbb{Q}}$. Then there is a collection of groups*

$$\mathcal{Q} = \{Q_p | G \leq_f Q_p \leq G^{\mathbb{Q}} \text{ and } |Q_p : G| = p^{\lambda_p} \text{ for each } p \in \mathcal{P}\}$$

with $Q_p = G$ for all but a finite number of groups Q_p such that

$$|T : G| = \prod_{p \in \mathcal{P}} p^{\lambda_p}.$$

Proof. We wish to form the collection $\mathcal{Q}$, that is we seek groups $G \leq_f Q_p \leq T$ such that $|Q_p : G| = p^{\lambda_p}$ for a finite number of primes $p \in \mathcal{P}$. For the other primes we then set $Q_p = G$. Therefore we seek Q_p for primes $p \mid |T : G|$.

It is clearly not always the case that $G \trianglelefteq T$ and so we set

$$C = \mathrm{Core}_T(G).$$

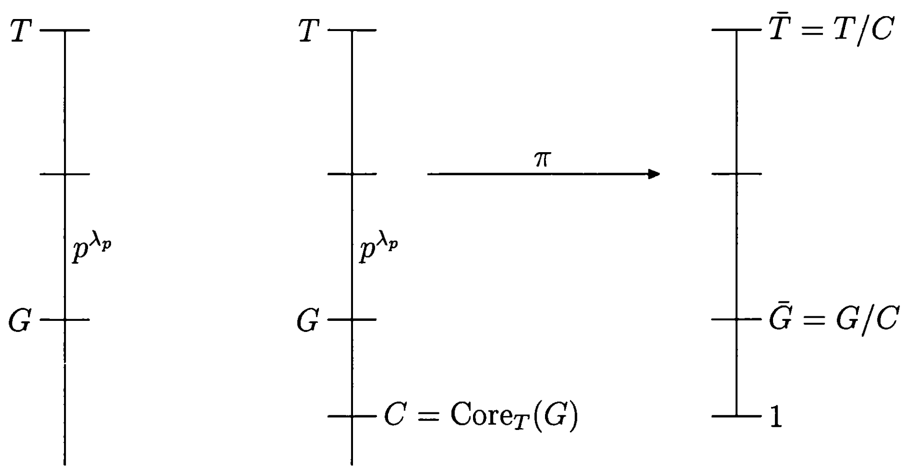


Figure 1.1: Projection of T to $T/\operatorname{Core}_T(G)$

Now $C \trianglelefteq_f T$ so that we may factor out C and work with finite nilpotent groups. Hence we form the projection $\pi : T \to T/C$ and for brevity denote the image under π of a subgroup $H \leq T$ as $\overline{H}$. Hence instead of seeking groups $G \leq_f Q_p \leq T$ with

$$p \mid |Q_p : G| \text{ and } p \not| |T : Q_p|$$

we seek groups $\overline{G} \leq_f \overline{L}_p \leq \overline{T}$ such that

$$p \mid |\overline{L}_p : \overline{G}| \text{ and } p \not| |\overline{T} : \overline{L}_p|.$$

The situation is summed up in Figure 1.1.

Let $p_1, p_2, \ldots, p_r$ be the primes involved in $|\overline{T}|$. Now $\overline{T}$ is a finite nilpotent group and hence we may write $\overline{T}$ as a Cartesian product of its Sylow p-subgroups,

$$\overline{T} = \overline{T}_1 \times \overline{T}_2 \times \cdots \times \overline{T}_r$$

and for each i, $\overline{T}_i$ is the Sylow p_i subgroup of $\overline{T}$. We do the same for $\overline{G}$ writing,

$$\overline{G} = \overline{G}_1 \times \overline{G}_2 \times \cdots \times \overline{G}_r$$

where each $\overline{G}_i$ is the Sylow p_i subgroup of $\overline{G}$. we note some of these factors may be trivial. Since $\overline{G} \leq \overline{T}$ we will have $\overline{G}_i \leq \overline{T}_i$. We note that $\overline{T}_{p_j} \trianglelefteq \overline{T}$ and so we may form the group

$$\overline{L}_k = \overline{G}_1 \times \overline{G}_2 \times \cdots \times \overline{T}_k \times \overline{G}_{k+1} \times \cdots \times \overline{G}_r.$$

By construction, each $\overline{L}_k$ is a group such that

$$|\overline{L}_k : \overline{G}| = p_k^{\mu_k}$$

for some $\mu_k \in \mathbb{N}$. We also have $p_k \not| |\overline{T} : \overline{L}_l|$ for $l \neq k$. Now

$$\langle \overline{L}_k | k = 1, 2, \ldots, r \rangle = \overline{T}$$

so that setting $Q_p = (\overline{L}_i)\pi^{-1}$ we have formed the desired collection. □

The two lemmas 1.9 and 1.10 show that we have an Euler product for the co-index zeta function of a $\mathcal{T}$-group, which we summarise below.

Theorem 1.11. *Let $G \in \mathcal{T}$ then*

$$\zeta_G^{up}(s) = \prod_{p \in \mathcal{P}} \zeta_G^{p,up}(s)$$

where

$$\zeta_G^{p,up}(s) = \sum_{\substack{G \leq_f T \leq G^{\mathbb{Q}} \\ |T:G| \text{ a } p\text{-power}}} |T : G|^{-s}.$$

1.6 Overview

In Chapter 2 we will go over a method for calculation of zeta functions of $\mathcal{T}$-groups. In Chapter 3 we will adapt this method to the co-index case and calculate some zeta functions.

In Chapters 4 and 5 we will look into other zeta and co-index zeta functions. In Chapter 6 we will apply these to look at the notion of sideways growth.

Chapter 2

Zeta functions of Groups

We now explain the good basis method for calculating zeta functions of $\mathcal{T}$-groups, as first given in in [GSS88]. We also calculate some "partial" zeta functions which we will use later on. Finally we calculate the zeta function of some finitely generated abelian groups by adapting a method of Smith [Smi83].

2.1 Good Bases

Let G be a poly-C_∞ group. Thus we have a series

$$1 = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_n = G \tag{2.1}$$

in which each factor $G_i/G_{i-1} \simeq C_\infty$. We may select $x_i \in G_i$ such that $\langle x_i G_{i-1} \rangle = G_i/G_{i-1}$. Now $G = \langle x_n, \ldots, x_1 \rangle$ and every element $y \in G$ is uniquely expressible as

$$y = x_n^{\lambda_n} x_{n-1}^{\lambda_{n-1}} \cdots x_1^{\lambda_1} = \underline{x}^{\underline{\lambda}}$$

for $\underline{\lambda} \in \mathbb{Z}^n$. We call $(x_n, \ldots, x_1)$ a canonical basis for G and the entries of $\underline{\lambda} = (\lambda_n, \ldots, \lambda_1)$ the parameters of y with respect to this basis.

If we fix the canonical basis we may identify G with the set $\mathbb{Z}^n$ by associating y with $\underline{\lambda}$ where $y = \underline{x}^{\underline{\lambda}}$. We place an ordering $\sqsubset$ on $\mathbb{Z}^n$ as follows:

(a) Order $\mathbb{Z}$ by $1 \prec 2 \prec \cdots \prec 0 \prec -1 \prec -2 \cdots$.

(b) Order $\mathbb{Z}^n$ lexicographically by $(\lambda_n, \lambda_{n-1}, \ldots, \lambda_1) \sqsubset (\mu_n, \mu_{n-1}, \ldots, \mu_1)$ *i.e.* if there is an i such that $\lambda_i \prec \mu_i$ but $\lambda_j = \mu_j$ for all $j > i$.

(This order $\prec$ of $\mathbb{Z}$ is well-founded and so the induced order $\sqsubset$ on $\mathbb{Z}^n$ is also well founded.)

Henceforth we will assume that G is a torsion-free finitely generated nilpotent group, so the series above can and will be chosen to be a refinement of the upper central series. In this context multiplication and exponentiation can be defined in terms of rational polynomials in $2n$ variables (*i.e* $\underline{\lambda}_1$ and $\underline{\lambda}_2$) and $n+1$ variables (*i.e* $\underline{\lambda}$ and the relevant power) respectively as shown in [Hal79].

Suppose G has a subgroup H of finite index (recall that we write this as $H \leq_f G$). Then as $G_i = \{(0, 0, \ldots, \gamma_i, \gamma_{i-1}, \ldots, \gamma_1) | \gamma_j \in \mathbb{Z}\}$, by the identification, then every element of $H_i = H \cap G_i$ is of the form

$$(0, 0, \ldots, \lambda_i, \lambda_{i-1}, \ldots, \lambda_1)$$

for suitable $\lambda_i, \lambda_{i-1}, \ldots \lambda_1 \in \mathbb{Z}$. We select

$$h_i = (0, \ldots, h_{i,i}, h_{i,(i-1)}, \ldots, h_{i,1}) \in H_i$$

to be minimal under $\sqsubset$ for each $i = 1, 2, \ldots, n$. Using the second isomorphism theorem, we have

$$\begin{aligned} H_i/H_{i-1} &= (G_i \cap H)/(G_{i-1} \cap H) \\ &= (G_i \cap H)/(G_{i-1} \cap (G_i \cap H)) \\ &\simeq (G_i \cap H)G_{i-1}/G_{i-1} \\ &= (G_i \cap HG_{i-1})/G_{i-1} \end{aligned}$$

and the final step is by the modular law. Now observe that,

$$(G_i \cap HG_{i-1})/G_{i-1} \leq G_i/G_{i-1}$$

so that H_i/H_{i-1} is cyclic, being isomorphic to a subgroup of a cyclic group. As G_i/G_{i-1} is generated by x_iG_{i-1} we have that

$$(G_i \cap HG_{i-1})/G_{i-1} = \langle x_i^\alpha G_{i-1} \rangle$$

for some $\alpha \in \mathbb{Z}$. Using the isomorphism,

$$\begin{array}{rlcl} \theta: & G_i \cap H/(G_{i-1} \cap (H \cap G_i)) & \rightarrow & (HG_{i-1} \cap G_i)/G_{i-1} \\ & y(G_{i-1} \cap (H \cap G_i)) & \mapsto & yG_{i-1} \end{array}$$

obtained above we find that H_i/H_{i-1} is generated by

$$(x_i^{\alpha} H_{i-1})\theta.^{-1}$$

We now wish to determine the generator of H_i/H_{i-1}.

Lemma 2.1. *The cyclic group H_i/H_{i-1} is generated by h_iH_{i-1}.*

Proof. The previous discussion has shown that H_i/H_{i-1} is indeed a cyclic group. Thus it remains to show that

$$\langle hH_{i-1} | h \in H_i \rangle = \langle h_i H_{i-1} \rangle.$$

Let $h \in H_i$ and then we can write

$$h = h_i^{\alpha_i} h_{i-1}^{\alpha_{i-1}} \cdots h_1^{\alpha_1}.$$

However,

$$h_{i-1}^{\alpha_{i-1}} h_{i-2}^{\alpha_{i-2}} \cdots h_1^{\alpha_1} \in H_{i-1}$$

so we have that

$$hH_{i-1} = h_i^{\alpha_i} h_{i-1}^{\alpha_{i-1}} \cdots h_1^{\alpha_1} H_{i-1} = h_i^{\alpha_i} H_{i-1}.$$

Thus,

$$\langle hH_{i-1} | h \in H_i \rangle = \langle h_i^{\alpha_i} H_{i-1} | \alpha_i \in \mathbb{Z} \rangle.$$

As h_i is minimal in H_i we have that

$$h_i \sqsubset h_i^{\alpha_i} \text{ for all } \alpha_i \in \mathbb{Z} - \{1\}$$

Thus

$$\langle h_i^{\alpha_i} | \alpha_i \in \mathbb{Z} \rangle = \langle h_i H_{i-1} \rangle$$

as required. □

Our aim now is to express $|G : H|$ using only information obtained from the

h_i. For this we use the following lemma (for a proof see [Lub93, Lemma 2.3], or observe that this is a case of the second isomorphism theorem),

Lemma. *If A and B are subgroups of C, with B normal in C, then $|C:A| = |B:A\cap B||C:AB|$.*

This allows us show the following lemma (which is folklore).

Lemma 2.2. *Let G and H be as above, then*

$$|G:H| = |G_n : H_nG_{n-1}||G_{n-1} : H_{n-1}G_{n-2}|\cdots|G_1 : H_1G_0|.$$

This is proved by the following process: firstly we observe that $|G:H| = |G_n : H_n|$ and then (in the notation of the above lemma) let $C = G_n$, $A = H_n$ and $B = G_{n-1}$ so that,

$$\begin{aligned} |G_n : H_n| &= |\ C \ : A \ \ B \ || \ B \ : A\cap B\ | \\ &= |\ G_n : H_n \ G_{n-1} \ || \ G_{n-1} : H_{n-1}\ |. \end{aligned}$$

Thus

$$|G_n : H_n| = |G_n : H_nG_{n-1}||G_{n-1} : H_{n-1}|,$$

and if we now let $C = G_{n-1}$, $A = H_{n-1}$ and $B = G_{n-2}$ we have,

$$\begin{aligned} |G_{n-1} : H_{n-1}| &= |\ C \ : A \ \ B \ || \ B \ : A\cap B\ | \\ &= |\ G_{n-1} : H_{n-1} \ G_{n-2} \ || \ G_{n-2} : H_{n-2}\ |. \end{aligned}$$

Hence,

$$|G_n : H_n| = |G_n : H_nG_{n-1}||G_{n-1} : H_{n-1}G_{n-2}||G_{n-2} : H_{n-2}|.$$

Continuing in this fashion, *i.e.* at each step j (for $j = 1, 2, \ldots, n$) we let $C = G_{n+1-j}$, $B = G_{n-j}$, and $A = H_{n+1-j}$, we obtain the stated result.

Clearly $H_iG_{i-1} \trianglelefteq G_i$ so that we may consider

$$|G_i/H_iG_{i-1}| = |\langle x_iH_iG_{i-1}\rangle| = h_{i,i}.$$

Hence we have that

$$|G:H| = \prod_{i=1}^{n} h_{i,i}.$$

Thus a finite index subgroup H of G gives rise to a sequence $h_n, h_{n-1}, \ldots, h_1$ with the following properties:

1. $|G : H| = \prod h_{i,i}$
2. $H_i = \langle h_i, h_{i-1}, \ldots, h_1 \rangle$ for $i = 1, \ldots n$.

Now we turn the problem around. Suppose instead of being given $H \leq_f G$, we are given a sequence $h_i \in G_i$ for $i = n, \ldots, 1$. We address the question of determining when there is a group $H \leq_f G$ which will give rise to this sequence. First we make the following definition.

Definition. *Let $h_i \in G_i$ for $i = 1, 2, \ldots, n$. Then h_i is* reduced *if*

1. $h_i \sqsubset (h_i)^{-1}$
2. $h_i \sqsubset h_i h_j^{\varepsilon}$ *for all $j < i$ and $\varepsilon \in \{1, -1\}$.*

The notion of being reduced is a condition on the exponents of the x_i in the basis elements h_i. As such, the condition that $h_n, h_{n-1} \ldots, h_1$ is reduced is like saying that the matrix of these coefficients is in Hermite Normal Form modulo the requirement that we must use the order $\sqsubset$ and only use integers.

Lemma 2.3. *Let G be a torsion-free nilpotent group and with central series (2.1) as above. Suppose that $h_i \in G_i - G_{i-1}$ for $i = 1, \ldots, n$. Define*

$$K_i = \{h_i^{\gamma_i} h_{i-1}^{\gamma_{i-1}} \cdots h_1^{\gamma_1} | \gamma_j \in \mathbb{Z} \text{ for } j = 1, \ldots, i\}.$$

Suppose further that $h_n, h_{n-1}, \ldots, h_1$ is reduced and for each i set K_i is a subgroup of G. Then

$$H = \langle h_n, \ldots, h_1 \rangle$$

has the property that $H_i = H \cap G_i = K_i$ and h_i is the minimal element of H_i.

Proof. We prove this in an number of steps. Firstly we show that K_i is the subgroup generated by $h_i, h_{i-1}, \ldots, h_1$. We then show h_i is minimal in K_i for each i. Finally we show $H_i = H \cap G_i = K_i$.

We observe that $h_j \in K_i$ for all $j \leq i$. Hence

$$\langle h_i, h_{i-1}, \ldots, h_1 \rangle \leq \langle K_i \rangle.$$

However K_i is a group (by assumption) and so $\langle h_i, h_{i-1}, \ldots, h_1 \rangle \leq K_i$. It is also clear that $K_i \leq \langle h_i, \ldots, h_1 \rangle$ hence we are done.

Now we show that h_i is minimal in K_i. Note that $h_1 = x_1^a$ for some $a \geq 0$ since h_1 is reduced (in the sense that the sequence $h_n, h_{n-1}, \ldots, h_1$ is reduced so that h_1 must be such that $h_1 \sqsubset h_1^{-1}$). Now $K_1 = \{x_1^{a\theta} | \theta \in \mathbb{Z}\}$ so that h_1 is minimal in K_1.

We proceed by induction, assuming that for $k < i$, we have h_k minimal in K_k and consider $h_i \in K_i$. This is minimal if

$$h_i = x_i^{h_{i,i}} x_{i-1}^{h_{i,(i-1)}} \cdots x_1^{h_{i,1}} \sqsubset g$$

for all $g \in K_i$. We assume to the contrary, so that $g \sqsubset h_i$ for some $g \in K_i$. Now

$$g = h_i^{\gamma_i} h_{i-1}^{\gamma_{i-1}} \cdots h_1^{\gamma_1} = x_i^{\theta_i} x_{i-1}^{\theta_{i-1}} \cdots x_1^{\theta_1}$$

for some $\theta_i \in \mathbb{Z}$. It must be the case that $\theta_i = h_{i,i}$. Hence

$$g = h_i h_{i-1}^{\gamma_{i-1}} \cdots h_1^{\gamma_1} = x_i^{h_{i,i}} x_{i-1}^{\theta_{i-1}} \cdots x_1^{\theta_1}$$

If $g \sqsubset h_i$, for some $1 \leq j \leq i$ we have:

1. $\theta_r = h_{i,r}$ for $i \geq r > i - j$
2. $\theta_{i-j} \prec h_{i,(i-j)}$.

However, we have assumed that $h_i, h_{i-1}, \ldots, h_1$ is reduced, so that $h_i \sqsubset h_i h_r^{\varepsilon}$ for all $1 \leq r < i$ and $\varepsilon \in \{1, -1\}$. So if $j = 1$ we have

$$\theta_{i-1} \prec h_{i,(i-1)}$$

and then it must be that $h_i h_{i-1}^{\varepsilon} \sqsubset h_i$. This contradicts our assumption that $h_i, h_{i-1}, \ldots, h_1$ is reduced. Hence, by induction, we have that h_i is minimal in K_i for all $i = 1, 2, \ldots, n$.

Finally we show that $H_i = H \cap G_i = K_i$. Firstly observe,

$$H_n = H = \langle h_n, \ldots, h_1 \rangle = K_n,$$

so $H_i = K_n \cap G_i$. Now $g \in G_i$ if

$$g = x_i^{a_i} x_{i-1}^{a_{i-1}} \cdots x_1^{a_1}.$$

If we also require $g \in K_n$, we must have,

$$g = h_n^{b_n} h_{n-1}^{b_{n-1}} \cdots h_1^{b_1}$$

and hence $g = h_i^{b_i} h_{i-1}^{b_{i-1}} \cdots h_1^{b_1}$ so that $H_i = K_i$ as required. $\square$

Thus we require that the given $h_n, h_{n-1}, \ldots, h_1$ are reduced and K_i is a group for $i = 1, \ldots, n$. We also need to to ensure that $|G : H|$ be finite. Firstly we investigate the conditions under which K_i is a group.

Lemma 2.4. *Let G be a torsion-free nilpotent group and with central series (2.1) as above. Suppose that $h_i \in G_i - G_{i-1}$ for $i = 1, \ldots, n$. Define*

$$K_i = \{h_i^{\gamma_i} h_{i-1}^{\gamma_{i-1}} \cdots h_1^{\gamma_1} | \gamma_j \in \mathbb{Z} \text{ for } j = 1, \ldots, i\}.$$

Suppose further that the sequence $h_n, h_{n-1}, \ldots, h_1$ is reduced. Then for each $i = 1, 2, \ldots, n$, K_i is a subgroup of G if and only if

$[h_i, h_k] \in K_{i-1}$ for all $k = 1, \ldots, i-1$.

Proof. Firstly, we assume that $[h_i, h_k] \in K_{i-1}$ for all $k = 1, \ldots, i-1$ and try to show that K_i is a group. We prove this by induction on i. We have that $h_1^{-1} \in K_1$ (by definition of K_1) so all we need to check is that $h_1^a h_1^b \in K_1$. However $h_1 = x_1^\theta$ so this is just

$$(x_1^\theta)^a (x_1^\theta)^b = (x_1^\theta)^{(a+b)} = h_1^{(a+b)}.$$

Hence K_1 is a group. Now let K_{i-1} be a group. We try to show that K_i is a subgroup of G. Thus we must check that if $g, h \in K_i$ then $gh \in K_i$ and $g^{-1} \in K_i$. Let

$$\begin{aligned} g &= h_i^{a_i} h_{i-1}^{a_{i-1}} \cdots h_1^{a_1} \\ h &= h_i^{b_i} h_{i-1}^{b_{i-1}} \cdots h_1^{b_1} \end{aligned}$$

then,

$$\begin{aligned} gh &= h_i^{a_i} h_{i-1}^{a_{i-1}} \cdots h_i^{b_i} h_{i-1}^{b_{i-1}} \cdots h_1^{b_1} \\ &= h_i^{a_i+b_i} h_{i-1}^{a_{i-1}} [h_i^{b_i}, h_{i-1}^{a_{i-1}}] h_{i-2}^{a_{i-2}} [h_i^{b_i}, h_{i-2}^{a_{i-2}}] \cdots h_1^{a_1} [h_i^{b_i}, h_1^{a_1}] h_{i-1}^{b_{i-1}} \cdots h_1^{b_1}. \end{aligned}$$

As K_{i-1} is a group and we have $[h_l, h_k] \in K_{i-1}$ for all $1 \leq k \leq i$ and $1 \leq l < k$. We can write this

$$h_i^{a_i+b_i} h_{i-1}^{c_{i-1}} \cdots h_1^{c_1}$$

so $gh \in K_i$ as required. As

It remains to show that $g^{-1} \in K_i$ for any $g \in K_i$. We note that by definition $h_i^{-1} \in K_i$. We now know that K_i is closed under multiplication. Since $g \in G$ we can write

$$\begin{aligned} g^{-1} &= (h_i^{a_i} h_{i-1}^{a_{i-1}} \cdots h_1^{a_1})^{-1} \\ &= h_1^{-a_1} h_2^{-a_2} \cdots h_i^{-a_i} \\ &= h_{i-1}^{c_{i-1}} h_{i-2}^{c_{i-2}} \cdots h_1^{c_1} h_i^{-a_i} \end{aligned}$$

for some $c_j \in \mathbb{Z}$, because K_{i-1} is a group. This is now a product of two elements of K_i and so is in K_i as K_i is closed under multiplication. Hence K_i is a group.

Conversely, we assume K_i is a subgroup of G. As G is nilpotent, so is K_i (as $K_i \leq G$) and hence if $K_i \neq 1$ we have $[K_i, K_i]$ is a proper subgroup of K_i. As the G_j have been chosen to be a refinement of the upper central series we have that $[K_i, K_i] \leq K_{i-1}$ and hence we have shown $[h_i, h_k] \in K_{i-1}$ for all $k = 1, \ldots, i-1$. $\square$

Finally we check the conditions which make H a subgroup of finite index in G. However we already know by the argument above that $|G : H| = \prod h_{i,i}$. Hence H is of finite index provided $h_{i,i} \neq 0$ for $i = 1, 2, \ldots, n$.

Consequently the conditions we need for a sequence $h_n, h_{n-1}, \ldots, h_1$ to define a subgroup H of finite index are:

1. $h_n, h_{n-1}, \ldots, h_1$ is reduced
2. $|G : H| = \prod h_{i,i}$
3. $H_i = \langle h_i, h_{i-1}, \ldots, h_1 \rangle$.

We now use this to calculate zeta function of certain nilpotent groups.

2.2 The Zeta Function of C_∞^d

The following result is classical and is included as a demonstration of the above method. Let $G = C_\infty^d = \langle x_d, x_{d-1}, \dots, x_1 \rangle$. As G is abelian we have that the upper central series for G is:

$$1 = G_0 \trianglelefteq G_1 = G$$

which we refine to

$$1 = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_d = G$$

by setting $G_i = \langle x_i, x_{i-1}, \dots, x_1 \rangle$ for $i = 1, 2, \dots, d$. We wish to count finite index subgroups H of G and so we seek valid bases of these subgroups as above. That is we seek $h_i \in G_i - G_{i-1}$ for $i = 1, \dots, d$ such that:

1. The sequence h_i is reduced
2. $K_i = \{h_i^{\alpha_i} h_{i-1}^{\alpha_{i-1}} \cdots h_1^{\alpha_1}\}$ is a group, that is, $[h_i, h_k] \in K_{i-1}$ for all $k = 1, \dots, i-1$.

We firstly observe that as G is abelian $[g, h] = 1$ for all $g, h \in G$, so that (2) is automatically satisfied. Hence we have that K_i is a group. Now recall that the sequence h_i is reduced if

(a) $h_i \sqsubset (h_i)^{-1}$

(b) $h_i \sqsubset h_i h_j^{\varepsilon}$ for all $j < i$ and $\varepsilon \in \{1, -1\}$.

We note that

$$h_i = x_i^{h_{i,i}} x_{i-1}^{h_{i,i-1}} \cdots x_1^{h_{i,1}}$$

and

$$(h_i)^{-1} = x_i^{-h_{i,i}} x_{i-1}^{-h_{i,i-1}} \cdots x_1^{-h_{i,1}}.$$

Thus if $h_i \sqsubset (h_i)^{-1}$ we must have that $h_{i,i} \prec -h_{i,i}$. In in the standard ordering of $\mathbb{Z}$ this means that $h_{i,i} > 0$ for (a) to hold.

We now consider (b); fix i and $j < i$. As G is abelian we have that,

$$\begin{aligned} h_i h_j^{\varepsilon} &= x_i^{h_{i,i}} x_{i-1}^{h_{i,i-1}} \cdots x_1^{h_{i,1}} \cdot x_j^{\varepsilon h_{j,j}} x_{j-1}^{\varepsilon h_{j,j-1}} \cdots x_1^{\varepsilon h_{j,1}} \\ &= x_i^{h_{i,i}} x_{i-1}^{h_{i,i-1}} \cdots x_j^{(h_{i,j}+\varepsilon h_{j,j})} x_{j-1}^{(h_{i,j-1}+\varepsilon h_{j,j-1})} \cdots x_1^{(h_{i,1}+\varepsilon h_{j,1})}. \end{aligned}$$

Thus if $h_i \sqsubset h_i h_j^{\varepsilon}$ it must be the case that

$$h_{i,j} \prec h_{i,j} \pm h_{j,j}. \tag{2.2}$$

We split this up into two cases:

1. $h_{i,j} \leq 0$
2. $h_{i,j} > 0$.

Firstly if $h_{i,j} \leq 0$ then for (2.2) to hold it must be the case that

$$h_{i,j} \pm h_{j,j} < h_{i,j}.$$

In other words we must have $h_{j,j} < 0$ and $h_{j,j} > 0$. This is a contradiction and so $h_{i,j} > 0$.

Thus as we require that (2.2) holds we must have $h_{i,j} > 0$ and $h_{j,j} > 0$. As we are dealing with positive integers it is now clear that

$$h_{i,j} \prec h_{i,j} + h_{j,j}.$$

So we must ensure that

$$h_{i,j} \prec h_{i,j} - h_{j,j}.$$

The only way that this can happen is if

$$h_{i,j} - h_{j,j} \leq 0.$$

Hence we have that that $h_{i,j} \leq h_{j,j}$. Therefore $h_d, h_{d-1}, \ldots, h_1$ are reduced if

1. $h_{i,i} > 0$ for $i = 1, 2, \ldots, d$
2. $0 < h_{i,j} \leq h_{j,j}$ for $i = 1, 2, \ldots, d$ and $j < i$.

We observe that the above process is a variant of finding the Hermite Normal form of a matrix. If we consider the good basis as a $d \times d$ matrix of coefficients, then our aim is to reduce this to the above form. The variations are in the fact that we wish to have integer leading coefficients, and we are not using the standard ordering of $\mathbb{Z}^n$. The order we have chosen has the property that we prefer small positive integers. This is not ideal but does mean that we can use a single order for comparison, and deal with basis elements not containing negative powers. An alternative consider later is to use a family of orderings, which after the first non-zero element of a row prefers zero, and then small positive integers. However, using this family of orders introduces complications and the analysis becomes more sensitive, so for the most part we will use the above order.

We now ask; how many choices of subgroups of this index are there? The basis we have formed for H is as follows:

$$\begin{array}{lclllll} h_d & = & h_d^{h_{d,d}} & h_{d-1}^{h_{d,d-1}} & \cdots & h_2^{h_{d,2}} & h_1^{h_{d,1}} \\ h_{d-1} & = & & h_{d-1}^{h_{d-1,d-1}} & \cdots & h_2^{h_{d-1,2}} & h_1^{h_{d-1,1}} \\ & \vdots & & & & \vdots & \\ h_2 & = & & & & h_2^{h_{2,2}} & h_1^{h_{2,1}} \\ h_1 & = & & & & & h_1^{h_{1,1}} \end{array}$$

with the above conditions. We know that this basis will form a subgroup H of G of index,

$$|G:H| = \prod_{i=1}^{d} h_{i,i}.$$

If we fix $h_{d,d}, h_{d-1,d-1}, \ldots, h_{1,1}$, we then have

$$\prod_{i=1}^{(d-1)} (h_{i,i})^{(d-i)}$$

choices of different valid bases for this subgroup H (that is we can pick $h_{i,j}$ to be $0 < h_{i,j} \leq h_{j,j}$ for $i = 2, \ldots, d$ and each choice yields a different good basis, and consequently a different subgroup (as the good basis elements are minimal)).

Thus,

$$\begin{aligned}
\zeta_{C_\infty^d}(s) &= \sum_{H\leq_f G} |G:H|^{-s} \\
&= \sum_{h_{d,d},h_{d-1,d-1},\ldots,h_{1,1}=1}^{\infty} \left(\prod_{i=1}^{d} h_{i,i}\right)^{-s}\left(\prod_{i=1}^{(d-1)} (h_{i,i})^{(d-i)}\right) \\
&= \sum_{h_{d,d},h_{d-1,d-1},\ldots,h_{1,1}=1}^{\infty} \left(\prod_{i=1}^{d} h_{i,i}^{-s+(d-i)}\right) \\
&= \zeta(s)\zeta(s-1)\cdots\zeta(s-d+1)
\end{aligned}$$

So we have shown the classical result

Theorem 2.5. *Let $G = C_\infty^d$ then*

$$\zeta_{C_\infty^d}(s) = \zeta(s)\zeta(s-1)\cdots\zeta(s-d+1).$$

There are other methods to calculate this for example, see [Lub93] for three different methods. We then have,

Corollary 2.6. *Let $G = C_\infty^d$ then $\alpha_G = d$.*

This follows because $\zeta(s)$ is the sum of $\frac{1}{s-1}$ and a holomorphic function when analytically continued to the left of the half plane $\mathrm{Re}(z) = 1$.

2.3 The Zeta Function of F_2^2

We now turn to the zeta function for F_2^2 the nilpotent group on 2 generators of class 2. This group, sometimes called the Heisenberg group, has presentation

$$H = \langle x, y, z | [x, y] = z, [x, z] = [y, z] = 1\rangle.$$

From [Smi83] we know that

Theorem 2.7. *[Smi83, Theorem 1.1]*

$$\zeta_{F_2^2}(s) = \frac{\zeta(s)\zeta(s-1)\zeta(2s-2)\zeta(2s-3)}{\zeta(3s-3)}.$$

Corollary 2.8. *$\alpha_{F_2^2} = 2$, as $\zeta_{F_2^2}(s)$ may be continued analytically left of the punctured line*

$$\{s|\,\mathrm{Re}(s) = 2\}/\{2\},$$

and at 2 $\zeta_{F_2^2}(s)$ has a double pole.

We briefly prove this using the good basis method described in §2.1 (and as in [Lub93]).

Proof. We note that the upper central series for $H = F_2^2$ is

$$1 \trianglelefteq \langle z \rangle \trianglelefteq H$$

and we refine this to the central series,

$$1 \trianglelefteq H_1 \trianglelefteq H_2 \trianglelefteq H_3 = H$$

where $H_1 = \langle z \rangle$ and $H_2 = \langle y, z \rangle$. We wish to count the subgroups of a given index and so we seek a good basis g_3, g_2, g_1 of a subgroup $G \leq_f H$. We recall that we must have

1. g_3, g_2, g_1 reduced
2. $G_3 = \{g_3^a g_2^b g_1^c | a, b, c \in \mathbb{Z}\}$, $G_2 = \{g_2^b g_1^c | b, c \in \mathbb{Z}\}$ and $G_1 = \{g_1^c | c \in \mathbb{Z}\}$ are groups, that is $[g_i, g_k] \in G_{k-1}$ for $k < i$.

Thus we select

$$\begin{array}{rcccc} g_3 & = & x^\alpha & y^\beta & z^\gamma \\ g_2 & = & & y^\delta & z^\varepsilon \\ g_1 & = & & & z^\lambda \end{array}$$

Now as $g_1 \in Z(H)$ it is clear that $[g_3, g_1] = [g_2, g_1] = 1$ so we only need check that $[g_3, g_2] \in G_2$, that is,

$$\begin{aligned} [x^\alpha y^\beta z^\gamma, y^\delta z^\varepsilon] &= [x^\alpha y^\beta, y^\delta] \text{ as } z \text{ is central} \\ &= [x^\alpha, y^\delta]^{y^\beta} [y^\beta, y^\delta] \\ &= z^{\alpha\delta} \cdot 1 \end{aligned}$$

where we have used one of P. Hall's commutator formula's (see [Hal79, Lemma 2.1] and Section 1.2 of the introduction).

Hence, if $z^{\alpha\delta} \in G_1$, we must have that $\alpha\delta = l\lambda$ for some $l \in \mathbb{Z}$ and we have shown condition 2. It remains to check that the sequence g_3, g_2, g_1 is reduced. Firstly we note that as we must have

$$g_i \sqsubset g_i^{-1}$$

for $i = 1, 2, 3$, it must be the case that $\alpha, \delta, \lambda \in \mathbb{N}$. Now if

$$g_2 \sqsubset g_2 g_1^{\pm 1} = y^{\delta} z^{\varepsilon \pm \lambda},$$

then we must have $\varepsilon \prec \varepsilon \pm \lambda$. As we already know that $\lambda \in \mathbb{N}$, we consider two cases, *i.e.* when $\varepsilon \leq 0$ and when $\varepsilon > 0$. If $\varepsilon \leq 0$, we must have

$$\begin{aligned} &\varepsilon \prec \varepsilon - \lambda \quad \text{and} \\ &\varepsilon \prec \varepsilon + \lambda \end{aligned}$$

that is

$$\begin{aligned} &\varepsilon - \lambda \leq \varepsilon \leq 0 \quad \text{and} \\ &\varepsilon + \lambda \leq \varepsilon \leq 0 \end{aligned}$$

both of which cannot occur. Hence $\varepsilon > 0$ and we need

$$\begin{aligned} &\varepsilon \leq \varepsilon + \lambda \quad \text{and} \\ &\varepsilon + \lambda \leq 0 \end{aligned}$$

for $\varepsilon \prec \varepsilon \pm \lambda$ to hold. Therefore we need $0 < \varepsilon \leq \lambda$.

Similar arguments will show us that we also need

$$\begin{aligned} &0 < \gamma \leq \lambda \\ &0 < \beta \leq \delta. \end{aligned}$$

Thus our good basis for a subgroup $G \leq_f H$ is

$$\begin{array}{rcccc} g_3 & = & x^{\alpha} & y^{\beta} & z^{\gamma} \\ g_2 & = & & y^{\delta} & z^{\varepsilon} \\ g_1 & = & & & z^{\lambda} \end{array}$$

with $\alpha, \beta, \gamma, \delta, \varepsilon, \lambda \in \mathbb{N}$, $0 < \beta \leq \delta$, $0 < \gamma, \varepsilon \leq \lambda$ and $\alpha\delta = l \cdot \lambda$ for some $l \in \mathbb{Z}$. Hence for a index of $\alpha\delta\lambda$ we have $\delta\lambda^2$ choices of groups (that is choices for good basis and each good basis yields a different subgroup due to the fact that t_i is minimal in T_i) of that index and thus,

$$\begin{aligned}\zeta_H(s) &= \sum_{\substack{\alpha\in\mathbb{N} \\ l\lambda=\alpha\delta}} \sum_{\substack{\delta\in\mathbb{N} \\ \text{for some } l}} \sum_{\lambda\in\mathbb{N}} (\alpha\delta\lambda)^{-s} \\ &= \frac{\zeta(s)\zeta(s-1)\zeta(2s-2)\zeta(2s-3)}{\zeta(3s-3)}.\end{aligned}$$

(Here, for brevity, we have omitted the finer details of the calculation.) □

2.4 The Zeta Functions of $\mathcal{H}$-groups

We now wish to continue by looking at other groups of Hirsch Length 3 and nil potency class 2 (which we call $\mathcal{H}$-groups). These groups can all be presented as

$$G = \langle x, y, z | [x, y] = z^t, [x, z] = [y, z] = 1\rangle$$

for some $t \in \mathbb{N}$, where if $t = 1$ we have $G = F_2^2$. As mentioned in the introduction we have $t = |Z(G) : \gamma_2(G)|$ and this classifies such a group up to isomorphism (see [Smi83]). For convenience we will call a group G as above H_t, and note the following result,

Theorem 2.9 ([Smi83, Theorem 1.4]). *Suppose $H_t \in \mathcal{H}$ and choose $p \in \mathcal{P}$. Suppose further that the p^{th}-part of t is p^θ then*

$$\zeta^p_{H_t}(s) = (1 - p^{\theta(2-s)})\zeta^p_{\mathbb{Z}^3}(s) + p^{\theta(2-s)}\zeta^p_H(s).$$

Proof. We apply the good basis method as for the Heisenberg group. We note that $H_t \in \mathcal{H}$ has upper central series $1 \trianglelefteq \langle z\rangle \trianglelefteq H_t$ and refine this to the series

$$1 \trianglelefteq H_{t,1} \trianglelefteq H_{t,2} \trianglelefteq H_{t,3} = H_t$$

where $H_{t,1} = \langle z\rangle$ and $H_{t,2} = \langle y, z\rangle$. We aim to count subgroups $G \leq_f H_t$ and so we pick a good basis

$$\begin{aligned} g_3 &= x^\alpha \quad y^\beta \quad z^\gamma \\ g_2 &= \qquad\; y^\delta \quad z^\varepsilon \\ g_1 &= \qquad\qquad\; z^\lambda \end{aligned}$$

with $g_3 \in G_3 - G_2$, $g_2 \in G_2 - G_1$ and $g_1 \in G_1 - \{1\}$. As we need $[g_3, g_2] \in G_2$ we must have

$$\begin{aligned}
[x^\alpha y^\beta z^\gamma, y^\delta z^\varepsilon] &= [x^\alpha y^\beta, y^\delta] \quad \text{as } z \text{ is central} \\
&= [x^\alpha, y^\delta]^{y^\beta}[y^\beta, y^\delta] \\
&= z^{t\alpha\delta}
\end{aligned}$$

so that we require $t\alpha\delta = l\lambda$ for some $l \in \mathbb{Z}$.

To determine when such a good basis is reduced follows a similar argument to that in the proof of the zeta function for F_2^2. We end up insisting that

$$\begin{array}{ccccc}
0 & < & \gamma, \varepsilon & \leq & \lambda \\
0 & < & \beta & \leq & \delta
\end{array}$$

with $\alpha, \beta, \gamma, \delta, \varepsilon, \lambda \in \mathbb{N}$. Thus we have,

$$\zeta_{H_t}(s) = \sum_{\substack{\alpha \in \mathbb{N}\ \delta \in \mathbb{N}\ \lambda \in \mathbb{N} \\ l\lambda = t\alpha\delta}} (\alpha\delta\lambda)^{-s}\delta\lambda^2.$$

If we reduce to the case of a single prime $p \in \mathcal{P}$, where the p^{th} part of t is p^θ, this sum becomes more tractable and we find that

$$\zeta_{H_t}^p(s) = (1 - p^{\theta(2-s)})\zeta_{\mathbb{Z}^3}^p(s) + p^{\theta(2-s)}\zeta_H^p(s).$$

(Again we have omitted the finer details of the tedious calculation.) $\square$

We now wish to estimate α_{H_t} for any $H_t \in \mathcal{H}$. We note, that for fixed $t \in \mathbb{N}$ there are only finitely many primes $p \in \mathcal{P}$ such that $p|t$. Thus we can partition $\mathcal{P} = \mathcal{D} \cup \mathcal{P} - \mathcal{D}$ where $\mathcal{D}$ contains all the prime divisors of t. We note that if $p \notin \mathcal{D}$ we have that

$$\zeta_{H_t}^p(s) = \zeta_H^p(s).$$

Hence we have,

$$\zeta_{H_t}(s) = \prod_{p \in \mathcal{P}} \zeta_{H_t}^p(s) = \prod_{p \in \mathcal{D}} \zeta_{H_t}^p(s) \prod_{p \in \mathcal{P} - \mathcal{D}} \zeta_H^p(s).$$

Now recall that

$$\zeta_H^p(s) = \frac{(1-p^{3-3s})}{(1-p^{-s})(1-p^{1-s})(1-p^{2-2s})(1-p^{3-2s})}$$

so that $p^{\theta(2-s)}\zeta_H^p(s)$ is a non-zero analytic functions for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 2$. Now

$$\zeta_{\mathbb{Z}^3}^p(s) = \frac{1}{(1-p^{-s})(1-p^{1-s})(1-p^{2-s})}$$

so that we have

$$(1-p^{\theta(2-s)})\zeta_{\mathbb{Z}^3}^p(s) = \frac{(1-p^{\theta(2-s)})}{(1-p^{2-s})}\zeta_{\mathbb{Z}^2}^p(s).$$

Now we observe that

$$\frac{(1-p^{\theta(2-s)})}{(1-p^{2-s})} = 1 + p^{(2-s)} + \cdots + p^{(\theta-1)(2-s)}$$

is an analytic functions for all $s \in \mathbb{C}$ and that $\zeta_{\mathbb{Z}^2}^p(s)$ is an analytic function for all s, with $\mathrm{Re}(s) > 2$. Therefore we have shown,

Lemma ([Smi83, Lemma 1.6]). *Let $H_t \in \mathcal{H}$ then*

$$\alpha_{H_t} = 2$$

with a double pole at $s = 2$.

2.5 Partial Zeta Functions

We now calculate a 'new' type of zeta function. We have already seen the standard zeta function of a group, however here we attempt to only look at those subgroups with a particular property. As we shall see later on, such properties can include whether the subgroups are normal in the original group, or if they are isomorphic to the original group. In this section we will adapt the latter point of view, however instead of just looking at the case when a subgroup is isomorphic to the original group, we wish to consider the case when it belongs to a specified isomorphism class. We will then compare these results with the (known) isomorphism zeta functions for certain groups.

If G is commensurable with F_2^2 then $G \in \mathcal{H}$. Thus if $L \leq_f F_2^2$, then $L \in \mathcal{H}$. So we may write

$$\zeta_{F_2^2}(s) = \sum_{G \in \mathcal{H}} \sum_{\substack{L \leq F_2^2 \\ G \simeq L}} |F_2^2 : L|^{-s}.$$

Hence we make the following definition,

Definition. *Let $H_r, H_t \in \mathcal{H}$ for some $r \in \mathbb{N}$. Define*

$$\phi_{H_t}^{H_r}(s) = \sum_{\substack{L \leq_f H_t \\ L \simeq H_r}} |H_t : L|^{-s}$$

We also set

$$\phi_{H_t}^{p,H_r}(s) = \sum_{\substack{L \leq_f H_t \\ L \simeq H_t \\ |H_t : L| \text{ a } p\text{-power}}} |H_t : L|^{-s}$$

(We make no claims of an Euler product for such functions.)

We now attempt to calculate $\phi_{H_r}^{H_t}(s)$ for firstly the case $t = 1$ and then when $t > 1$. To do this we will use the following lemma,

Lemma 2.10. *Let $H_r, H_t \in \mathcal{H}$. If $L \leq_f H_t = \langle x, y, z | [x, y] = z^t, [x, z] = [y, z] = 1 \rangle$ with good basis*

$$\begin{array}{ccc} x^{\alpha} & y^{\beta} & z^{\gamma} \\ & y^{\delta} & z^{\varepsilon} \\ & & z^{\lambda} \end{array}$$

then $L \simeq H_r$ if $t\alpha\delta = r\lambda$.

Proof. Let $H_r = \langle u, v, w | [u, v] = w^r, [u, w] = [v, w] = 1 \rangle$ and define a map $\theta : H_r \to L$ as follows

$$\begin{aligned} u &\mapsto x^{\alpha} y^{\beta} z^{\gamma} \\ v &\mapsto y^{\delta} z^{\varepsilon} \\ w &\mapsto z^{\lambda}. \end{aligned}$$

Thus as θ maps a basis of H_r to a basis of L this will be an isomorphism if it is a homomorphism. Hence we must check that

$$([u, v])\theta = (w^r)\theta.$$

Now,

$$\begin{aligned}([u,v])\theta &= (x^\alpha y^\beta z^\gamma)^{-1}(y^\delta z^\varepsilon)^{-1}(x^\alpha y^\beta z^\gamma)(y^\delta z^\varepsilon) \\ &= (x^{-\alpha}y^{-\beta}z^{-\gamma-t\alpha\beta})(y^{-\delta}z^{-\varepsilon})(x^\alpha y^{(\beta+\delta)}z^{\gamma+\varepsilon})\end{aligned}$$

(recalling that in H_t we have that

$$\begin{aligned}x^a y^b z^c \cdot x^d y^e z^f &= x^{a+d}y^{b+e}z^{c+f-tdb} \\ (x^a y^b z^c)^{-1} &= x^{-a}y^{-b}z^{-c-tab}). \\ (x^a y^b z^c)^n &= x^{na}y^{nb}z^{nc-t\binom{n}{2}ab}).\end{aligned}$$

Hence,

$$\begin{aligned}([u,v])\theta &= (x^{-\alpha}y^{-\beta-\delta}z^{-\gamma-\varepsilon-t\alpha\beta})(x^\alpha y^{(\beta+\delta)}z^{\gamma+\varepsilon}) \\ &= (x^{-\alpha}y^{-(\beta+\delta)}z^{-\gamma-\varepsilon-t\alpha\beta})(y^{(\beta+\delta)}x^\alpha z^{t\alpha(\beta+\delta)}z^{\gamma+\varepsilon}) \\ &= z^{-t\alpha\beta+t\alpha(\beta+\delta)}.\end{aligned}$$

Now we require that

$$(w^r)\theta = ([u,v])\theta = z^{t\alpha\delta}$$

and

$$(w^r)\theta = (z^\lambda)^r$$

Thus we must have $t\alpha\delta = r\lambda$ if $L \simeq H_r$. □

We are now in a position to calculate $\phi_H^{p,H_t}(s)$.

Proposition 2.11. *Let H be the Heisenberg group, so $H = \langle x,y,z|[x,y] = z, z$ central $\rangle$ then,*

$$\phi_H^{p,H_t}(s) = \frac{p^{-\theta s}\zeta^p(2s-2)}{(1-p)} - \frac{pp^{\theta(1-s)}\zeta^p(2s-3)}{(1-p)}$$

where θ is the p^{th} part of t for some prime $p \in \mathcal{P}$.

Before we give the proof of this we note that if $t = 1$ then we have (from Proposition 2.11)

$$\begin{aligned}
\phi_H^{p,H}(s) &= \frac{\zeta^p(2s-2)}{(1-p)} - \frac{p\zeta^p(2s-3)}{(1-p)} \\
&= \frac{1}{(1-p)}\left[\zeta^p(2s-2) - p\zeta^p(2s-3)\right] \\
&= \frac{1}{(1-p)}\left[\frac{1}{(1-p^{2-2s})} - \frac{p}{(1-p^{3-2s})}\right] \\
&= \frac{1}{(1-p)}\left[\frac{(1-p^{3-2s}) - (p-p^{3-2s})}{(1-p^{2-2s})(1-p^{3-2s})}\right] \\
&= \frac{(1-p)}{(1-p)}\left[\frac{1}{(1-p^{2-2s})(1-p^{3-2s})}\right] \\
&= \zeta^p(2s-2)\zeta^p(2s-3).
\end{aligned}$$

This, as expected, is the isomorphism zeta function for the Heisenberg group, see section 5.1.

Proof. If $G \leq H$ then we know from above that G is given by a good basis

$$\begin{array}{ccc}
x^a & y^b & z^c \\
 & y^d & z^e \\
 & & z^f
\end{array}$$

with $a, b, c, d, e, f \in \mathbb{N}$ such that,

$$\begin{gathered}
lf = ad \text{ for some } l \in \mathbb{N} \\
0 < b \leq d \\
0 < c, e \leq f.
\end{gathered}$$

It thus remains to calculate when $G \simeq H_t$ for some fixed t. This calculation is performed in Lemma 2.10, and we conclude that we need $tf = ad$. This condition does not lead to an as tractable calculation of $\phi_H^{H_t}(s)$ as we would wish, so for ease we calculate for a given prime $p \in \mathcal{P}$. Then we have, as for H,

$$\phi_H^{p,H_t}(s) = \sum_{\substack{a=0 \\ \\ a+b=c+\theta}}^{\infty}\sum_{b=0}^{\infty}\sum_{c=0}^{\infty} (p^a p^b p^c)^{-s} p^b p^{2c}$$

where p^θ is the p$^{\text{th}}$-part of t (which is $\theta = 0$ for all but a finite number of primes

$p \in \mathcal{P}$). We now observe that the condition $a + b = c + \theta$ can be replaced by the condition that $a + b \geq \theta$ provided we set $c = a + b - \theta$. So,

$$\sum_{\substack{a=0 \\ a+b=c+\theta}}^{\infty} \sum_{b=0}^{\infty} \sum_{c=0}^{\infty} (p^a p^b p^c)^{-s} p^b p^{2c} = \sum_{\substack{a=0 \\ a+b\geq\theta, c=a+b-\theta}}^{\infty} \sum_{b=0}^{\infty} (p^a p^b p^{a+b-\theta})^{-s} p^b p^{2(a+b-\theta)}.$$

Hence we have,

$$\begin{aligned}
\phi_H^{p,H_t}(s) &= p^{\theta(s-2)} \sum_{\substack{a=0 \\ a+b\geq\theta}}^{\infty} \sum_{b=0}^{\infty} (p^{2a} p^{2b})^{-s} p^{3b} p^{2a} \\
&= p^{\theta(s-2)} \left(\sum_{a=0}^{\infty} \sum_{b=0}^{\infty} (p^{2a} p^{2b})^{-s} p^{3b} p^{2a} - \sum_{\substack{a=0 \\ a+b<\theta}}^{\infty} \sum_{b=0}^{\infty} (p^{2a} p^{2b})^{-s} p^{3b} p^{2a} \right).
\end{aligned}$$

In this last step we have summed over all a, b and subtracted those which fail to meet the condition $a + b \geq \theta$. If we now let $i = a + b$ so we have $0 \leq b \leq i < \theta$, and the above becomes

$$\begin{aligned}
\phi_H^{p,H_t}(s) &= p^{\theta(s-2)} \left(\sum_{a=0}^{\infty} \sum_{b=0}^{\infty} (p^{2a} p^{2b})^{-s} p^{3b} p^{2a} - \sum_{i=0}^{\theta-1} \sum_{b=0}^{i} (p^{2i})^{-s} p^b p^{2i} \right) \\
&= p^{\theta(s-2)} \left(\zeta^p(2s-2)\zeta^p(2s-3) - \sum_{i=0}^{\theta-1} \sum_{b=0}^{i} (p^{2i})^{-s} p^b p^{2i} \right). \qquad (2.3)
\end{aligned}$$

If we now consider the last term in the above (2.3), and change it so that we now sum over all b we have,

$$\begin{aligned}
\sum_{i=0}^{\theta-1} \sum_{b=0}^{i} (p^{2i})^{-s} p^b p^{2i} &= \sum_{i=0}^{\theta-1} \sum_{b=0}^{\infty} (p^{2i})^{-s} p^b p^{2i} - \sum_{i=0}^{\theta-1} \sum_{b=i+1}^{\infty} (p^{2i})^{-s} p^b p^{2i} \\
&= \sum_{i=0}^{\theta-1} \sum_{b=0}^{\infty} (p^{2i})^{-s} p^b p^{2i} - \sum_{i=0}^{\theta-1} \sum_{c=0}^{\infty} (p^{2i})^{-s} p^{c+i+1} p^{2i} \\
&= \sum_{i=0}^{\theta-1} \sum_{b=0}^{\infty} (p^{2i})^{-s} p^b p^{2i} - p \cdot \sum_{i=0}^{\theta-1} \sum_{c=0}^{\infty} (p^{2i})^{-s} p^c p^{3i} \qquad (2.4)
\end{aligned}$$

where we have set $b = i + 1 + c$. We now consider the last term in (2.4). We sum over all valid i by considering all possible i and subtracting invalid values,

$$
\begin{aligned}
p \cdot \sum_{i=0}^{\theta-1} \sum_{c=0}^{\infty} (p^{2i})^{-s} p^c p^{3i} &= p \left(\sum_{i=0}^{\infty} \sum_{c=0}^{\infty} (p^{2i})^{-s} p^c p^{3i} - \sum_{i=\theta}^{\infty} \sum_{c=0}^{\infty} (p^{2i})^{-s} p^c p^{3i} \right) \\
&= p \left(\frac{\zeta^p(2s-3)}{(1-p)} - \sum_{j=0}^{\infty} \sum_{c=0}^{\infty} (p^{2(j+\theta)})^{-s} p^c p^{3(j+\theta)} \right) \\
&= p \left(\frac{\zeta^p(2s-3)}{(1-p)} - \sum_{j=0}^{\infty} \sum_{c=0}^{\infty} p^{\theta(3-2s)} p^{j(3-2s)} p^c \right) \\
&= p \left(\frac{\zeta^p(2s-3)}{(1-p)} - p^{\theta(3-2s)} \frac{\zeta^p(2s-3)}{(1-p)} \right)
\end{aligned}
$$

where we have set $i = j + \theta$. We deal with the first term of (2.4) in a similar fashion,

$$
\begin{aligned}
\sum_{i=0}^{\theta-1} \sum_{b=0}^{\infty} (p^{2i})^{-s} p^b p^{2i} &= \sum_{i=0}^{\infty} \sum_{b=0}^{\infty} (p^{2i})^{-s} p^b p^{2i} - \sum_{i=\theta}^{\infty} \sum_{b=0}^{\infty} (p^{2i})^{-s} p^b p^{2i} \\
&= \frac{\zeta^p(2s-2)}{(1-p)} - \sum_{j=0}^{\infty} \sum_{b=0}^{\infty} (p^{2(j+\theta)})^{-s} p^b p^{2(j+\theta)} \\
&= \frac{\zeta^p(2s-2)}{(1-p)} - p^{\theta(2-2s)} \frac{\zeta^p(2s-2)}{(1-p)}.
\end{aligned}
$$

Thus combining the above we see that (2.3) becomes,

$$
\begin{aligned}
\phi_H^{p,H_t}(s) &= p^{\theta(s-2)} \Bigg(\zeta^p(2s-2)\zeta^p(2s-3) - \Bigg(\frac{\zeta^p(2s-2)}{(1-p)} - \\
&\quad p^{\theta(2-2s)} \frac{\zeta^p(2s-2)}{(1-p)} - p \left(\frac{\zeta^p(2s-3)}{(1-p)} - p^{\theta(3-2s)} \frac{\zeta^p(2s-3)}{(1-p)} \right) \Bigg) \Bigg) \\
&= p^{\theta(s-2)} \Bigg(\zeta^p(2s-2)\zeta^p(2s-3) - \frac{\zeta^p(2s-2}{(1-p)} + p \frac{\zeta^p(2s-3)}{(1-p)} + \\
&\quad p^{\theta(2-2s)} \frac{\zeta^p(2s-2)}{(1-p)} - p p^{\theta(3-2s)} \frac{\zeta^p(2s-3)}{(1-p)} \Bigg).
\end{aligned}
$$

Now observing that

$$
\begin{aligned}
\frac{1}{(1-p)} \left(p\zeta^p(2s-3) - \zeta^p(2s-2) \right) &= \frac{1}{(1-p)} \left(\frac{p}{(1-p^{3-2s})} - \frac{1}{(1-p^{2-2s})} \right) \\
&= \frac{1}{(1-p)} \left(\frac{(p-p^{3-2s}) - (1-p^{3-2s})}{(1-p^{2-2s})(1-p^{3-2s})} \right) \\
&= \frac{-(1-p)}{(1-p)} \zeta^p(2s-2)\zeta^p(2s-3)
\end{aligned}
$$

we have

$$\begin{aligned}
\phi_H^{p,H_t}(s) &= p^{\theta(s-2)}\left(p^{\theta(2-2s)}\frac{\zeta^p(2s-2)}{(1-p)} - pp^{\theta(3-2s)}\frac{\zeta^p(2s-3)}{(1-p)}\right) \\
&= p^{-\theta s}\frac{\zeta^p(2s-2)}{(1-p)} - pp^{\theta(1-s)}\frac{\zeta^p(2s-3)}{(1-p)}
\end{aligned}$$

as required. □

We now check that this is indeed the case as we know that

$$\zeta_H^p(s) = \sum_{t\in\mathbb{N}} \phi_H^{p,H_t}(s).$$

Hence,

$$\begin{aligned}
\zeta_H^p(s) &= \sum_{t\in\mathbb{N}}\left(\frac{p^{-\theta s}\zeta^p(2s-2)}{(1-p)} - \frac{pp^{\theta(1-s)}\zeta^p(2s-3)}{(1-p)}\right) \\
&= \frac{\zeta^p(s)\zeta^p(2s-2)}{(1-p)} - \frac{p\zeta^p(s-1)\zeta^p(2s-3)}{(1-p)} \\
&= \frac{1}{(1-p)}\left[\frac{1}{(1-p^{-s})(1-p^{2-2s})} - \frac{p}{(1-p^{1-s})(1-p^{3-2s})}\right] \\
&= \frac{1}{(1-p)}\left[\frac{(1-p^{1-s}-p^{3-2s}+p^{4-3s})-(p-p^{1-s}-p^{3-2s}+p^{3-3s})}{(1-p^{-s})(1-p^{2-2s})(1-p^{1-s})(1-p^{3-2s})}\right] \\
&= \frac{\zeta^p(s)\zeta^p(s-1)\zeta^p(2s-2)\zeta^p(2s-3)}{(1-p)}\left[1-p-p^{3-3s}+p^{4-3s}\right] \\
&= \frac{\zeta^p(s)\zeta^p(s-1)\zeta^p(2s-2)\zeta^p(2s-3)}{(1-p)}\cdot(1-p)(1-p^{3-3s}) \\
&= \frac{\zeta^p(s)\zeta^p(s-1)\zeta^p(2s-2)\zeta^p(2s-3)}{\zeta^p(3s-3)}
\end{aligned}$$

as expected.

We can also using Lemma 2.10 calculate,

Proposition 2.12. *Let $H_r, H_t \in \mathcal{H}$ for some $r, t \in \mathbb{N}$ and let the p^{th}-part of r be p^ρ and the p^{th} -part of t be p^θ. Then if $\theta - \rho \leq 0$*

$$\phi_{H_t}^{p,H_r}(s) = \frac{p^{\theta(2-s)}}{p^{\rho(2-s)}}\zeta^p(2s-2)\zeta^p(2s-3)$$

and if $\theta - \rho < 0$

$$\phi_{H_t}^{p,H_r}(s) = \frac{p^{\theta(2-s)}}{p^{\rho(2-s)}} \left(\frac{\zeta^p(2s-2)}{(1-p)} - \frac{pp^{\theta}\zeta^p(2s-3)}{p^{\rho}(1-p)} \right).$$

Proof. (Sketch) The proof is similar to Proposition 2.11, and so we omit most of it. As before we begin by picking a good basis for H_t and impose the conditions of Lemma 2.10 so we end up (in the notation of Proposition 2.11) with the condition that $t\alpha\delta = r\lambda$. Moving to the case for a prime $p \in \mathcal{P}$ we see that

$$\phi_{H_t}^{p,H_r}(s) = \sum_{\substack{\alpha\in\mathbb{N} \\ \lambda=\theta-\rho+\alpha+\delta}} \sum_{\delta\in\mathbb{N}} \sum_{\lambda\in\mathbb{N}} (p^{\alpha}p^{\delta}p^{\lambda})^{-s}p^{2\lambda}p^{\delta}$$

Thus as we must have $\lambda \leq 0$ we split the sum into two cases, when $\theta - \rho \leq 0$ and $\theta - \rho < 0$. Using methods similar to the above, we can then simplify this sum to the stated results. □

We also observe that if we have $\frac{t}{r} \in \mathbb{N}$ we have no need to use the above "prime" version and we have,

Lemma 2.13. *Let* $H_r, H_t \in \mathcal{H}$ *for some* $r, t \in \mathbb{N}$. *Suppose that* $\frac{t}{r} \in \mathbb{N}$ *then*

$$\phi_{H_t}^{H_r}(s) = \left(\frac{t}{r}\right)^{2-s} \zeta(2s-2)\zeta(2s-3).$$

2.6 Zeta Functions for some Abelian Groups

We will now calculate some zeta functions of some abelian groups with torsion. We do this utilising an inductive method for showing $\zeta_G(s)$ when $G = C_\infty^d$ first given in [Smi83]. The following results do not seem to be contained in the literature, however from [McD97] we know that

$$\zeta_{C_2\times C_\infty^d}(s) = (1 + 2^{d-s})\zeta_{C_\infty^d}(s).$$

We aim to prove similar results for groups of the form $M \times C_\infty^d$ where M is a finite abelian group. The link to the crystallographic groups is by way of the following theorem;

Theorem. *[dSMS99, Theorem 1.3] There exist two plane crystallographic groups with non-isomorphic pro-finite completions but with the same number of subgroups of each finite index.*

The two groups in question are

$$\begin{aligned} \mathbf{p1} &= \langle x, y | [x, y] \rangle \simeq C_\infty^2 \\ \mathbf{pg} &= \langle x, y, t | [x, y], t^2 = y, x^t = x^{-1} \rangle \end{aligned}$$

and their shared zeta function is

$$\zeta_{\mathbf{p1}}(s) = \zeta_{\mathbf{pg}}(s) = \zeta(s)\zeta(s-1).$$

However there are also two families of such groups (*i.e.* having non-isomorphic pro-finite completions) which share zeta functions. One of them is

$$\{C_2 \times C_\infty^d | d \in \mathbb{N}, d \geq 2\},$$

and the other is

$$\begin{aligned} \{\Gamma_n = \langle t, x_1, \ldots, x_n | t^2, x_1^t = x_1, x_i^t = x_i^{-1}, \\ [x_j, x_k] \text{ for } 1 < i \leq n, 1 \leq j < k \leq n \rangle | n \in \mathbb{N}, n \geq 2\}, \end{aligned}$$

which is the analogue in n dimensions of the plane crystallographic group $\mathbf{pm}$ (for details of this see [McD97, Section 7.4]).

We now say something of the classical theory of derivations which we will use to calculate the zeta functions of $M \times C_\infty^d$. Let G be a group with a normal abelian subgroup N. Conjugation of G on N induces an action of G/N on N via $n^{Nx} = n^x$. Using this we define a derivation $\delta : G/N \to N$ to be a map δ such that $(ab)\delta = (a\delta)^b(b\delta)$. Note that if G is abelian (and this is the case we are interested in) then this map is just a homomorphism.

Suppose further that $L \leq G$ with $LN = G$ and $L \cap N = 1$. We define δ_L :

$G/N \to N$ by $(hN)\delta_L = n$ where $h = ln$. We observe that δ_L is a derivation:

$$\begin{aligned}(h_1Nh_2N)\delta_L = (h_1h_2N)\delta_L = (l_1l_2n_1^{l_2}n_2N)\delta_L &= n_1^{l_2}n_2 \\ &= n_2^{-1}n_1^{l_2}n_2n_2 \text{ as } N \text{ is abelian} \\ &= n_1^{h_2}n_2.\end{aligned}$$

Now let $\delta : G/N \to N$ be a derivation and $H \leq G$ with $HN = G$ and $H \cap N = 1$. Define $\psi_\delta : H \to G$ by $h \mapsto h(hN)\delta$. We claim that this map is then a monomorphism for:

$$\begin{aligned}(h_1h_2)\psi_\delta = h_1h_2(h_1h_2N)\delta &= h_1h_2(h_1N)\delta_2^h(h_2N)\delta \\ &= h_1(h_1N)\delta h_2(h_2N)\delta\end{aligned}$$

and $(h)\psi_\delta = 1$ if and only if $h \in N$. Defining $L = \text{Im } \psi_\delta$ we observe that $L \simeq H$ so that $G = LN$ and $L \cap N = 1$.

The two maps $\alpha : \delta \mapsto \text{Im } \psi_\delta$ and $\beta : L \mapsto \delta_L$ then set up a bijection between derivations $G/N \to N$ and the set of complements for N in G. This method is behind the proofs of the following results of Smith [Smi83].

Lemma. *[Smi83, Lemma 1.3] Let G be a group, $G = A \times B$ and $Q \leq B \leq L \leq G$ then there exists a $H \leq G$ such that $HB = L$ and $H \cap B = Q$.*

Lemma. *[Smi83, Lemma 1.4] Subject to the conditions above and with the further proviso that $Q \trianglelefteq G$ is abelian, the number of possible choices for $H \leq G$ is $|Der(L/B, B/Q)|$. (Note that $Der(A, B) = \{\delta : A \to B | \delta \text{ is a derivation}\}$.)*

(Note the slightly stronger (correct) condition in this last lemma than in [Smi83]).

In [Smi83] these results are used to give an inductive method to calculate $\zeta_{C_\infty^d}(s)$. We first of all generalise the result above for $C_2 \times C_\infty^d$ to any prime $p \in \mathcal{P}$.

Theorem 2.14. *Let $G = C_p \times C_\infty^d$ for some prime $p \in \mathcal{P}$. Then*

$$\zeta_G(s) = (1 + p^{d-s})\zeta_{C_\infty^d}(s).$$

Proof. In the notation of the above lemmas, let $A = C_\infty^d$ and $B = C_p$. Thus we have that $Q \leq B = C_p$ is one of $Q = C_p$ or $Q = 1$. We count subgroups of G by enumerating all pairs (L, Q) with $B \leq L \leq G$, and counting the resulting groups

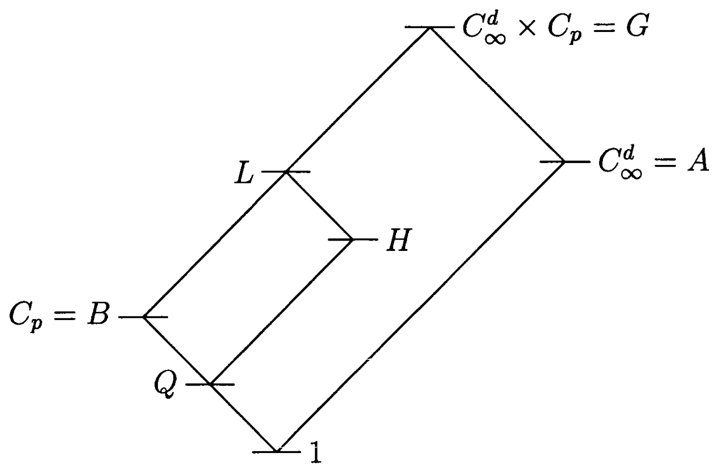


Figure 2.1: Subgroups in the proof of Theorem 2.14

H with $HB = L$ and $H \cap B = Q$ as above. So we have a situation as in the Figure 2.1.

Let

$$\zeta_{C_\infty^d}(s) = \sum_{n \in \mathbb{N}} a_n n^{-s}$$

so that a_n is the number of subgroups of index n in C_∞^d. We also note that

$$|G : H| = |G : L||L : H|$$

and that, by the second isomorphism theorem, we have

$$|L : H| = |C_p : Q|.$$

Hence $|L : H| = 1$ or $|L : H| = p$.

Let $|G : H| = m$. We consider two cases, either $p \nmid m$ or $p|m$. If $p \nmid m$ then in particular $p \nmid |L : H|$ so that we must have $|C_p : Q| = 1$. Thus $Q = C_p$ and the number of H is

$$|\operatorname{Der}(L/B, B/Q)| = |\operatorname{Der}(L/B, 1)| = 1.$$

If $p|m$ we have that $Q = C_p$ or $Q = 1$. Hence in addition to the above case we must consider the case when $Q = 1$. Here we have

$$|\operatorname{Der}(L/B, B/Q)| = |\operatorname{Der}(L/B, C_p)| = |\operatorname{Hom}(C_\infty^d, C_p)| = p^d$$

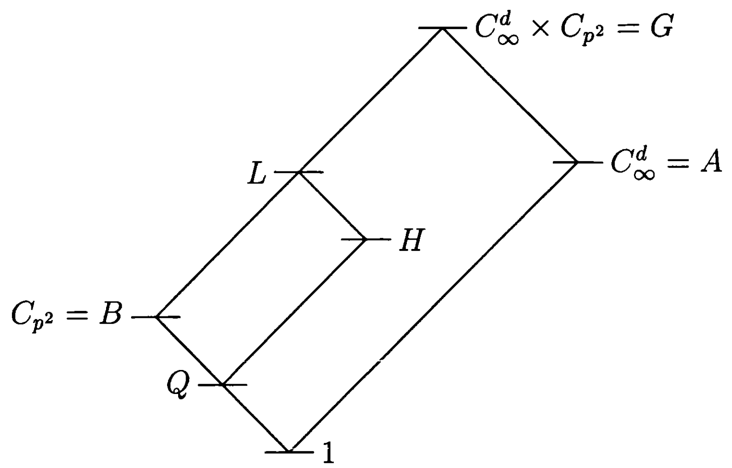


Figure 2.2: Subgroups in the proof of Theorem 2.15

where this is valid as both $L/B \simeq C_\infty^d$ and $B/Q \simeq C_p$ are abelian. Thus each (L, Q) pair (where $Q = 1$) gives rise to p^d choices for H.

Thus we have

$$\zeta_G(s) = \sum_{n \in \mathbb{N}} a_n n^{-s} + \sum_{n \in \mathbb{N}} a_n p^d (np)^{-s}$$

where the first summand is the case when $Q = C_p$ and the second is when $Q = 1$. Hence,

$$\begin{aligned} \zeta_G(s) &= \sum_{n \in \mathbb{N}} a_n n^{-s} + p^d p^{-s} \sum_{n \in \mathbb{N}} a_n n^{-s} \\ &= (1 + p^{d-s}) \sum_{n \in \mathbb{N}} a_n n^{-s} \\ &= (1 + p^{d-s}) \zeta_{C_\infty^d}(s). \end{aligned}$$

□

Clearly we can take this further,

Theorem 2.15. *Let $G = C_\infty^d \times C_{p^2}$ for some prime $p \in \mathcal{P}$. Then*

$$\zeta_G(s) = (1 + p^{d-s} + p^{2(d-s)}) \zeta_{C_\infty^d}.$$

Proof. We proceed as in Theorem 2.14. That is we let $A = C_\infty^d$ and $B = C_{p^2}$, and seek pairs (L, Q) with $B \leq L \leq G$, $Q \leq B$ such that $HB = L$ and $H \cap B = Q$. So we have the situation as in Figure 2.2.

Let

$$\zeta_{C_\infty^d}(s) = \sum_{n\in\mathbb{N}} a_n n^{-s}$$

so that a_n is the number of subgroups of index n in C_∞^d and note that

$$|G:H| = |G:L||L:H| = |G:L||B:Q|.$$

If $Q \leq B$ then Q is isomorphic to one of 1, C_p or C_{p^2}. We again let $|G:H| = m$ and consider the three cases $p \not| m$, $p \mid m$ and $p^2 \mid m$.

If $p \nmid m$ then it must be the case that $Q = B$ so that $B/Q \simeq 1$ hence we have a single choice of H. If $p \mid m$ then in addition to the above case we must consider the single group $Q \leq B$ with $Q \simeq C_p$. Thus we have

$$|\operatorname{Der}(L/B, B/Q)| = |\operatorname{Der}(C_\infty^d, C_{p^2})| = |\operatorname{Hom}(C_\infty^d, C_{p^2})| = p^d$$

choices for H. Finally if $p^2 \mid m$ then we could also have the case $Q = 1$ so that we have

$$|\operatorname{Der}(L/B, B/Q)| = |\operatorname{Der}(C_\infty^d, C_p)| = |\operatorname{Hom}(C_\infty^d, C_p)| = p^{2d}.$$

Hence we have p^{2d} choices for H in this case. Thus

$$\begin{aligned}\zeta_G(s) &= \sum_{n\in\mathbb{N}} a_n n^{-s} + \sum_{n\in\mathbb{N}} a_n p^d (np)^{-s} \sum_{n\in\mathbb{N}} a_n p^{2d}(np^2)^{-s} \\ &= (1 + p^{(d-s)} + p^{2(d-s)})\zeta_{C_\infty^d}(s).\end{aligned}$$

□

In fact, we can generalise this to the following,

Theorem 2.16. *Let $G = C_\infty^d \times C_{p^r}$ for some prime $p \in \mathcal{P}$ and $r \in \mathbb{N}$. Then*

$$\zeta_G(s) = \left(\sum_{i=0}^{r} p^{i(d-s)}\right)\zeta_{C_\infty^d}(s).$$

(We omit the proof as it is similar to the above.)

Following on from this, the next step is clearly, to examine what happens when the torsion subgroup is not elementary abelian.

Theorem 2.17. *Let $p, q \in \mathcal{P}$ with $p \neq q$ and let $G = C_p \times C_q \times C_\infty^d$ then*

$$\zeta_G(s) = (1 + q^d p^{-s})(1 + p^d q^{-s})\zeta_{C_\infty^d}(s).$$

Proof. The proof follows the approach to that of Theorem 2.15. Let $A = C_\infty^d$ and $B = C_p \times C_q$. Thus if $Q \leq B$ then we must have Q being one of $1, C_p, C_q$ or $C_p \times C_q$. We count subgroups of G by enumerating all pairs (L, Q) and counting the resulting groups H such that $HB = L$ and $H \cap B = Q$. Let

$$\zeta_{C_\infty^d}(s) = \sum_{n \in \mathbb{N}} a_n n^{-s}$$

so that a_n is the number of subgroups of index n in C_∞^d. We also note that $|G : H| = |G : L||L : H|$ and that we have $|L : H| = |C_p \times C_q : Q|$. Hence we have one of,

1. $|L : H| = 1$
2. $|L : H| = p$
3. $|L : H| = q$
4. $|L : H| = pq$.

Let $|G : H| = m$ and we consider the four cases as above. Hence if $p \nmid m$ and $q \nmid m$ we must have $|L : H| = 1$ and so $Q = C_q \times C_p$. The number of possible H is then,

$$|\operatorname{Der}(L/B, B/Q)| = |\operatorname{Der}(L/B, 1)| = 1$$

and hence for each L there is a single H.

If we have $p|m$ and $q \nmid m$, then we can have either $Q = C_q$ or $Q = C_q \times C_p$. Thus in addition to the above we must consider what happens when $Q = C_q$. If $Q = C_q$, we have that $B/Q = C_p$ and so we have

$$|\operatorname{Der}(L/B, B/Q)| = |\operatorname{Der}(C_\infty^d, C_p)| = |\operatorname{Hom}(C_\infty^d, C_p)| = p^d$$

possible H. Similarly, if $q|m$ and $p \nmid m$, the we have q^d possible choices for H.

Finally if both $p|m$ and $q|m$ we must consider, in addition to the above, the case when $Q = 1$, so that $B/Q = C_p \times C_q$. Thus we have

$$|\operatorname{Der}(C_\infty^d, C_p)| = |\operatorname{Hom}(C_\infty^d, C_p \times C_q)| = (pq)^d$$

choices for H. Hence, putting this altogether we see that

$$\begin{aligned}
\zeta_G(s) &= \sum_{n\in\mathbb{N}} a_n n^{-s} + \sum_{n\in\mathbb{N}} a_n(q^d)(pn)^{-s} + \sum_{n\in\mathbb{N}} a_n(p^d)(qn)^{-s} + \sum_{n\in\mathbb{N}} a_n(pq)^d(pqn)^{-s} \\
&= (1 + q^d p^{-s} + p^d + q^{-s} + (pd)^{d-s})\zeta_{C_\infty^d}(s) \\
&= (1 + q^d p^{-s})(1 + p^d q^{-s})\zeta_{C_\infty^d}(s).
\end{aligned}$$

□

We note that in the above we have only considered cyclic groups, or groups where there has been a single subgroup of each valid index. We now move away from this using the following lemma.

Lemma 2.18. *Let $G = \operatorname{Cart}_{i=1}^r C_{p^m}$ for some prime $p \in \mathcal{P}$ and fixed $m \in \mathbb{N}$. Let*

$$\zeta_{C_\infty^r}^p(s) = \sum_{j=0}^{\infty} b_j p^{-js}$$

then the number of subgroups of index p^k for $k = 0, 1, \ldots \lfloor \frac{mr}{2} \rfloor$ is b_k.

The Duality theorem for finite abelian groups (see page 63 for more details) then allows us to find the number of subgroups of all indices.

Proof. We have an epimorphism $\theta : C_\infty^r \to G$ given by

$$x_i \mapsto g_i$$

where $C_\infty^r = \langle x_1, x_2, \ldots, x_r \rangle$ and each factor C_{p^m} of G is generated by g_i. This map then sets up a bijection between subgroups of C_∞^r above the kernel of θ (we note that $\operatorname{Ker} C_\infty^r = \langle x_1^{p^m}, x_2^{p^m}, \ldots, x_r^{p^m} \rangle$) and subgroups of G giving us the stated result. □

Thus we can prove

Theorem 2.19. *Let $M = \mathrm{Cart}_{i=1}^{r} C_{p^m}$ for some prime $p \in \mathcal{P}$ and fixed $m \in \mathbb{N}$. Let $G = M \times C_\infty^d$, then*

$$\zeta_G(s) = \left(\sum_{l=0}^{rm} b_l p^{m-l} p^l \right) \zeta_{C_\infty^d}(s).$$

where b_l is the number of subgroup of index p^l in M.

Proof. We again use the above derivation style argument, which we will now simply sketch. Let $A = C_\infty^d$ and $B = M$. We count subgroups by enumerating pairs (L, Q) with $B \leq L \leq Q$ and counting the resulting groups H with $H \cap B = Q$. We note that

$$|G : H| = |G : L||L : H| \text{ and } |L : H| = |B : Q|.$$

Thus we must consider when $p^l \mid |G : H|$ and $p^{l+1} \nmid |G : H|$ for each $l = 0, 1, \ldots r$. Each case includes the previous case. If we fix l, we then have b_l subgroups of index p^m and size p^{m-l}. Thus we have

$$|\operatorname{Der}(L/B, B/Q)| = |\operatorname{Hom}(C_\infty^d, B/Q)| = p^{m-l}$$

as C_∞^d and B/Q are abelian. This will happen for each l and hence,

$$\zeta_G(s) = \left(\sum_{l=0}^{rm} b_l p^{m-l} p^l \right) \zeta_{C_\infty^d}(s).$$

□

At this point we stop our calculations, as this is not the main thrust of this thesis. We note however that these calculations could be carried on, or even performed using a method similar to that in [McD97].

Chapter 3

Co-index zeta functions of $\mathcal{T}$-groups

We now attempt to calculate the co-index zeta functions of some $\mathcal{T}$-groups. To do this we first give another construction of the Mal'cev completion $G^{\mathbb{Q}}$ of a $\mathcal{T}$-group G. We then use this construction to form a good basis for a finite co-index overgroup $T \leq G^{\mathbb{Q}}$ of G before using this good basis method to calculate some co-index zeta functions.

3.1 The Mal'cev Completion

We begin to build the Mal'cev completion of a $\mathcal{T}$-group G. We go about this in a way such that we can then start to form good bases of overgroups of a group within this completion in an analogous way to that of good bases for subgroups. Let G be a torsion free finitely-generated nilpotent group of class c. Such a group will have an upper central series

$$1 = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_c = G$$

which may be refined to a central series

$$1 = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_n = G \tag{3.1}$$

in which each factor $G_i/G_{i-1} \simeq C_\infty$. This means that we may select $x_i \in G_i - G_{i-1}$ such that $G_i = \langle G_{i-1}, x_i \rangle$. Thus every element $y \in G$ is uniquely expressible as

$$y = x_n^{\alpha_n} x_{n-1}^{\alpha_{n-1}} \cdots x_1^{\alpha_1} = \underline{x}^{\underline{\alpha}}$$

for $\underline{\alpha} \in \mathbb{Z}^n$. Now let $w = \underline{x}^{\underline{\beta}}$ be another element of G and consider the product wy. Now $wy \in G$ and so we can write

$$wy = x_n^{\gamma_n} x_{n-1}^{\gamma_{n-1}} \cdots x_1^{\gamma_1} = \underline{x}^{\underline{\gamma}}.$$

Each γ_i is determined by the $2n$ integer variables α_i, β_i and so there is a function $f_i : \mathbb{Z}^{2n} \to \mathbb{Z}$ such that $\gamma_i = f_i(\underline{\alpha}, \underline{\beta})$. In a similar way if $\lambda \in \mathbb{Z}$ we have that

$$w^\lambda = x_n^{\delta_n} x_{n-1}^{\delta_{n-1}} \cdots x_1^{\delta_1} = \underline{x}^{\underline{\delta}}$$

so that δ_i depends only on $\underline{\alpha}$ and λ Thus we have a function $g_i : \mathbb{Z}^{n+1} \to \mathbb{Z}$ where $\delta_i = g_i(\underline{\alpha}, \lambda)$.

As shown in [Hal79, Theorem 6.5] these functions are defined by rational polynomials. These polynomials are integer valued (in that an integer argument will yield an integer value). As a polynomial is determined by its values at integral integer arguments, there is no ambiguity if we identify the polynomials with the function. Furthermore a polynomial which is integer valued can be written as a $\mathbb{Z}$-linear combination of binomial polynomials with integer coefficients. That is, we may write each polynomial as a $\mathbb{Z}$-linear combination of products of the form,

$$\binom{\alpha_1}{r_1}\binom{\alpha_2}{r_2}\cdots\binom{\alpha_n}{r_n}\binom{\beta_1}{s_1}\binom{\beta_2}{s_2}\cdots\binom{\beta_n}{s_n}$$

where r_j and s_k are integers greater than or equal to zero.

Such polynomials, when given rational arguments, will take values that lie in $\mathbb{Q}$. Using this, we use the polynomials to define a group $G^{\mathbb{Q}}$; $G^{\mathbb{Q}}$ will consist of all elements $\underline{x}^{\underline{\gamma}}$ with $\underline{\gamma} \in \mathbb{Q}$. Multiplication and exponentiation are defined in terms of the above polynomials. The associative law is then given by a polynomial identity with the other group axioms following easily. We also have the embedding $\iota : G \to G^{\mathbb{Q}}$ which allows us to identify G with $\iota(G) \leq G^{\mathbb{Q}}$.

We call $G^{\mathbb{Q}}$ the Mal'cev completion of G and observe that unlike the construction in Section 1.3 the properties of $G^{\mathbb{Q}}$ are not so easy to "read off." However, as we know that any two Mal'cev completions are isomorphic we can observe the following properties (which are clear from the constructions given in the introduction):

1. The group $G^{\mathbb{Q}}$ is radicable (*i.e.* if $m \in \mathbb{N}$ and $g \in G^{\mathbb{Q}}$, then there is a $h \in G^{\mathbb{Q}}$ such that $h^m = g$. In fact such an h is unique).

2. The group $G^{\mathbb{Q}} = I_{G^{\mathbb{Q}}}(G) = \{g \in G^{\mathbb{Q}} | g^n \in G \text{ for some } n \in \mathbb{N}\}$.

3. The group $G^{\mathbb{Q}}$ is torsion free. (For if $w^{\lambda} = 1$ for $1 \neq w \in G^{\mathbb{Q}}$ and $\lambda \in \mathbb{N}$, then $w \in G^{\mathbb{Q}} - G$ as G is torsion free. Further, if we pick λ to be the smallest possible value such that $w^{\lambda} = 1$ we must have that $w^c \notin G$ for $1 \leq C < \lambda$ as G is torsion free. However given $w \in G^{\mathbb{Q}}$ there is a $g \in G$ such that $w^m = g \neq 1$ for some $m \in \mathbb{N}$. This is a contradiction and so $G^{\mathbb{Q}}$ is torsion free.)

4. The group $G^{\mathbb{Q}}$ has the same nilpotency class as G ($G^{\mathbb{Q}}$ is certainly of class at most c (the nilpotency class of G) as $\gamma_{c+1}(G^{\mathbb{Q}})$ is generated by commutators of weight $c+1$, which can be given by a polynomial identity. However as $G \leq G^{\mathbb{Q}}$ the class of G is bounded by the class of $G^{\mathbb{Q}}$ so it must be that $G^{\mathbb{Q}}$ is of nilpotency class c.)

As shown in the introduction, see Section 1.3, we have that if H is a radicable torsion free nilpotent group then if $\eta : G \to H$ is a homomorphism then η extends uniquely to a homomorphism $\eta^{\mathbb{Q}} : G^{\mathbb{Q}} \to H$. This property allows us to show that $G^{\mathbb{Q}}$ is unique up to isomorphism, in the sense that it is characterised by the above properties. However, we omit this here and observe that the above construction is independent of the choice of the x_i above.

3.1.1 Good bases for overgroups

We wish to count finite co-index overgroups of G which are contained in $G^{\mathbb{Q}}$, i.e groups T such that

$$G \leq_f T \leq G^{\mathbb{Q}}.$$

Note that we are concerned with overgroups contained in a fixed Mal'cev completion of G. Henceforth we will omit this remark, but the provision will be in force throughout, unless noted otherwise. We recall the definition of the following formal sum, the co-index zeta function of G

$$\zeta_G^{\mathrm{up}}(s) = \sum_{G \leq_f T \leq G^{\mathbb{Q}}} |T : G|^{-s},$$

and approach the calculation of this by finding bases for these overgroups.

We place an ordering $\sqsubset_{\mathbb{Q}}$ on $\mathbb{Q}^n$ as follows:

(a) Firstly order $\mathbb{Q}$ is the usual manner (denoted by $<$). For $a, b \in \mathbb{Q}$ we write $a \prec_{\mathbb{Q}} b$ if

   i) $0 < a, b$ and $a < b$, or

   ii) $0 < a$ and $b < 0$, or

   iii) $a, b \leq 0$ and $b < a$

   (in other words we prefer small positive numbers to large positive numbers, then negative numbers of small modulus followed by ones of large modulus, and finally zero).

(b) Order $\mathbb{Q}^n$ lexicographically by $(\lambda_n, \lambda_{n-1}, \dots, \lambda_1) \sqsubset_{\mathbb{Q}} (\mu_n, \mu_{n-1}, \dots, \mu_1)$ if there is an i such that $\lambda_i \prec_{\mathbb{Q}} \mu_i$ but $\lambda_j = \mu_j$ for all $j > i$.

We identify $G^{\mathbb{Q}}$ with $\mathbb{Q}^n$ by associating $\underline{\lambda} = (\lambda_n, \lambda_{n-1}, \dots, \lambda_1)$ to

$$y = x_n^{\lambda_n} x_{n-1}^{\lambda_{n-1}} \cdots x_1^{\lambda_1} = \underline{x}^{\underline{\lambda}}.$$

We now fix the above basis for G and using this form $G^{\mathbb{Q}}$ its Mal'cev completion. As observed above we can write

$$G^{\mathbb{Q}} = \{(\gamma_n, \gamma_{n-1}, \dots, \gamma_1) | \gamma_j \in \mathbb{Q} \text{ for } j = 1, 2, \dots, n\}.$$

We now wish to have a central series of $G^{\mathbb{Q}}$; to do this we form the Mal'cev completions of the members G_j for $j = 1, 2, \dots, n$, of the refinement of the upper central series (3.1). As in these groups multiplication and exponentiation are defined as in $G^{\mathbb{Q}}$ we find that

$$1 = G_0^{\mathbb{Q}} \trianglelefteq G_1^{\mathbb{Q}} \trianglelefteq \cdots \trianglelefteq G_n^{\mathbb{Q}} = G^{\mathbb{Q}}$$

is a central series of $G^{\mathbb{Q}}$ with

$$G_i^{\mathbb{Q}} = \{(\gamma_n, \gamma_{n-1}, \dots, \gamma_1) | \gamma_j \in \mathbb{Q} \quad \forall j \text{ and } \gamma_n = \gamma_{n-1} = \dots = \gamma_{i+1} = 0\}$$

and thus $G_i^{\mathbb{Q}}/G_{i-1}^{\mathbb{Q}} \simeq \mathbb{Q}$ for all i.

Let T be an overgroup of G of finite co-index (that is $G \leq_f T$). We define

$$T_i = T \cap G_i^{\mathbb{Q}}.$$

We consider

$$T_i/T_{i-1} = \langle tT_{i-1} | t \in T_i \rangle.$$

Now, by definition we have,

$$T_i/T_{i-1} = (T \cap G_i^{\mathbb{Q}})/(T \cap G_{i-1}^{\mathbb{Q}})$$

and we may write this as $(T \cap G_i^{\mathbb{Q}})/(T \cap G_i^{\mathbb{Q}} \cap G_{i-1}^{\mathbb{Q}})$ so that the second isomorphism theorem applies and we have

$$(T \cap G_i^{\mathbb{Q}})/(T \cap G_i^{\mathbb{Q}} \cap G_{i-1}^{\mathbb{Q}}) \simeq (T \cap G_i^{\mathbb{Q}})G_{i-1}^{\mathbb{Q}}/G_{i-1}^{\mathbb{Q}} \leq G_i^{\mathbb{Q}}/G_{i-1}^{\mathbb{Q}}.$$

Now, $G_i^{\mathbb{Q}}/G_{i-1}^{\mathbb{Q}} \simeq (\mathbb{Q}, +)$ and $(T \cap G_i^{\mathbb{Q}})G_{i-1}^{\mathbb{Q}}/G_{i-1}^{\mathbb{Q}}$ is finitely generated. Hence we have that T_i/T_{i-1} is cyclic, as any finitely generated subgroup of $(\mathbb{Q}, +)$ is cyclic.

We now select, for $i = 1, 2, \dots, n$,

$$t_i \in T_i - T_{i-1} = T \cap I_{\mathbb{Q}}(G_i)$$

to be minimal under $\sqsubset_{\mathbb{Q}}$ and let

$$t_i = (0, 0, \dots, 0, t_{i,i}, t_{i,i-1}, \dots, t_{i,1}).$$

We wish to show now that

$$T_i/T_{i-1} = \langle t_i T_{i-1} \rangle$$

Firstly we note that

$$T_1/T_0 = \langle x_1^{\alpha} \rangle$$

for some $\alpha \in \mathbb{Q}$ (as this is a cyclic group). Since t_1 is minimal subject to $\sqsubset_{\mathbb{Q}}$ we must have that $t_1 = x_1^{\alpha}$. Now consider, T_i/T_{i-1}. If $t \in T_i$ then we have $t = t_i^{\alpha_i} t_{i-1}^{\alpha_{i-1}} \cdots t_1^{\alpha_1}$ by construction. As $t_{i-1}^{\alpha_{i-1}} \cdots t_1^{\alpha_1} \in T_{i-1}$ we have that $tT_{i-1} =$

$t_i^{\alpha_i}T_{i-1}$. Thus

$$T_i/T_{i-1} = \langle t_i^{\alpha}T_{i-1} | \alpha \in \mathbb{Z}\rangle.$$

Hence as t_i is minimal under $\sqsubset_{\mathbb{Q}}$ we have that $T_i/T_{i-1} = \langle t_iT_{i-1}\rangle$ as required.

We now define what it means to be reduced in our current setting,

Definition. *Let $t_i \in T_i$ for $i = 1, 2, \ldots, n$. Then t_i is* reduced *if*

1. $t_i \sqsubset_{\mathbb{Q}} (t_i)^{-1}$
2. $t_i \sqsubset_{\mathbb{Q}} t_it_j^{\varepsilon}$ *for all $j < i$ and $\varepsilon \in \{1, -1\}$.*

We now reuse [Lub93, Lemma 2.3]. This allowed us to write, for a subgroup $H \leq_f G$ of a $\mathcal{T}$-group G, with central series $1 = G_0 \trianglelefteq G_1 \trianglelefteq \cdots G_n = G$,

$$|G : H| = |G_n : H_nG_{n-1}||G_{n-1} : H_{n-1}G_{n-2}| \cdots |G_1 : H_1G_0|.$$

where $H_i = G_i \cap H$. If we replace G by T and H by G we find that,

$$|T : G| = |T_n : G_nT_{n-1}||T_{n-1} : G_{n-1}T_{n-2}| \cdots |T_1 : G_1T_0|.$$

It now remains to calculate $|T_j : G_jT_{j-1}|$ for $j = 1, 2, \ldots n$. We firstly prove,

Lemma 3.1. *Let G be a torsion-free finitely generated nilpotent group with central series as above. Let $G \leq_f T \leq G^{\mathbb{Q}}$. If t_i is defined as above for $i = 1, 2, \ldots, n$ we find that*

$$t_{i,i} = \alpha_i^{-1}$$

where $\alpha_i \in \mathbb{N}$.

Proof. We recall that

$$T_i = T \cap G_i^{\mathbb{Q}}.$$

We also note from the above discussion that T_i/T_{i-1} is generated by t_iT_{i-1} with t_i minimal under $\sqsubset_{\mathbb{Q}}$. Hence, as $t_i = (0, 0, \ldots, t_{i,i}, t_{i,i-1}, \ldots t_{i,1})$ we will have that $t_{i,i} > 0$. Now $G_i \leq T_i$ and so we have $x_i \in T_i$. Thus it must be possible to write

$$x_i = t_i^{\alpha_i}t_{i-1}^{\alpha_{i-1}} \cdots t_1^{\alpha_1}$$

for some $\alpha_i \in \mathbb{Z}$. As $t_i = x_i^{t_{i,i}}x_{i-1}^{t_{i,i-1}} \cdots x_1^{t_{i,1}}$ and x_i does not occur in any other t_j

for $j < i$ it suffices to show that for some $\alpha \in \mathbb{Z}$,

$$t_i^\alpha = x_i \cdot (x_{i-1}^{\beta_{i-1}} \cdots x_1^{\beta_1})$$

with $\beta_j \in \mathbb{Q}$ for $j = 1, 2, \ldots, i-1$. So we write,

$$\begin{aligned} t_i^\alpha &= (x_i^{t_{i,i}} x_{i-1}^{t_{i,i-1}} \cdots x_1^{t_{i,1}})^\alpha \\ &= (x^{t_{i,i}})^\alpha \cdot y \end{aligned}$$

where $y \in G_{i-1}^{\mathbb{Q}}$. We can write this because y will only involve $x_{i-1}, x_{i-1}, \ldots x_1$ and commutators of the form $[x_i, x_j]$ which must lie in T_{i-1} since T is nilpotent and the T_i form a central series.

Therefore $t_{i,i}\alpha = 1$ and so $t_{i,i} = \alpha^{-1}$. As t_i is minimal under $\sqsubset_{\mathbb{Q}}$ we will have that $t_{i,i} > 0$ so that $\alpha \in \mathbb{N}$. $\square$

We now calculate $|T_i : G_i T_{i-1}|$. We have that $G_i T_{i-1} \trianglelefteq T_i$ as certainly $T_{i-1} \trianglelefteq T_n$ and if $x \in G_i$ and $y \in T_i$ we have,

$$x^y = x \cdot [x, y].$$

As T is nilpotent $[x, y] \in T_{i-1}$ so that $x \cdot [x, y] \in G_i T_{i-1}$. Now,

$$T_i / G_i T_{i-1} = \langle t G_i T_{i-1} | t \in T_i \rangle$$

and as we know if $t \in T_i$,

$$t = t_i^{\alpha_i} t_{i-1}^{\alpha_{i-1}} \cdots t_1^{\alpha_1}$$

with $t_{i-1}^{\alpha_{i-1}} \cdots t_1^{\alpha_1} \in T_{i-1}$ we have

$$\langle t G_i T_{i-1} | t \in T_i \rangle = \langle t_i^\alpha G_i T_{i-1} \rangle.$$

We also have that

$$t_i = x_i^{t_{i,i}} x_{i-1}^{t_{i,i-1}} \cdots x_1^{t_{i,1}}$$

with $t_{i,i} = \frac{1}{\beta}$, say. So that

$$t_i^\beta G_i T_{i-1} = x_i G_i T_{i-1}$$

as $t_i^\beta y = x_i$ for some $y \in T_{i-1}$. Thus $|T_i : G_i T_{i-1}|$ is at most β. If $0 < \alpha < \beta$ we have that from a central series

$$t_i^{\alpha} = x_i^{\alpha t_{i,1}} \cdot y$$

with $y \in G_{i-1}^{\mathbb{Q}}$. As $x_i^{\alpha t_{i,1}} \notin T_{i-1}$ and as $\alpha < \beta$ we have $x_i^{\alpha t_{i,1}} \notin G_i$ we find that

$$|T_i : G_i T_{i-1}| = \beta = t_{i,i}^{-1}.$$

Hence,

$$\begin{aligned} |T : G| &= |T_n : G_n T_{n-1}||T_{n-1} : G_{n-1} T_{n-2}| \cdots |T_1 : G_1 T_0| \\ &= \prod_{i=1}^{n} t_{i,i}^{-1}. \end{aligned}$$

Thus a finite co-index group T of G gives rise to a sequence $t_n, t_{n-1}, \ldots, t_1$ with the following properties:

1. $|T : G| = \prod t_{i,i}^{-1}$
2. $G_i = \langle t_i, t_{i-1}, \ldots, t_1 \rangle$ for $i = 1, \ldots n$

Now we turn the problem around. Suppose instead of being given an overgroup T of G of finite co-index, we are given a sequence $t_i \in G_i^{\mathbb{Q}} - G_{i-1}^{\mathbb{Q}}$ for $i = 1, 2, \ldots, n$. We address the question of determining when there is a group T such that $G \leq_f T$ which will give rise to this sequence. The first two lemmas are sufficiently similar to the subgroup case that we omit their proofs.

Lemma 3.2. *Let G be a torsion-free nilpotent group and with central series (3.1) as above. Suppose that $t_i \in G_i^{\mathbb{Q}} - G_{i-1}^{\mathbb{Q}}$ for $i = 1, \ldots, n$. Define*

$$L_i = \{t_i^{\gamma_i} t_{i-1}^{\gamma_{i-1}} \cdots t_1^{\gamma_1} | \gamma_j \in \mathbb{Z} \text{ for } j = 1, \ldots, i\}.$$

Suppose further that the sequence t_i is reduced and the set L_i is a subgroup of $G^{\mathbb{Q}}$. Then

$$T = \langle t_n, \ldots, t_1 \rangle$$

has the property that $T_i = T \cap G_i^{\mathbb{Q}} = L_i$ and t_i is the minimal element of T_i.

Lemma 3.3. *Let G be a torsion-free nilpotent group and with central series (3.1)* *as above. Suppose that $t_i \in G_i^{\mathbb{Q}} - G_{i-1}^{\mathbb{Q}}$ for $i = 1, \dots, n$. Define*

$$T_i = \{t_i^{\gamma_i} t_{i-1}^{\gamma_{i-1}} \cdots t_1^{\gamma_1} | \gamma_j \in \mathbb{Z} \textit{ for } j = 1, \dots, i\}.$$

Suppose further that the sequence t_i is reduced. Then for each $i = 1, 2, \dots, n$, T_i is a subgroup of $G^{\mathbb{Q}}$ if and only if $[t_i, t_k] \in T_{i-1}$ for all $k = 1, \dots, i-1$.

Note that we have not yet addressed the question of how to enforce the fact that $G \leq_f T$. We must impose some extra conditions to do this. The basic condition we need is that $x_i \in T_i$. However we may express this as follows,

Lemma 3.4. *Let G and $t_n, t_{n-1}, \dots, t_1$ be as above such that $T = \langle t_n, t_{n-1}, \dots, t_1 \rangle$ is a subgroup of $G^{\mathbb{Q}}$. If*

(1.) $t_{i,i} = \alpha_i^{-1}$ for all i and $\alpha_i \in \mathbb{Z}$

(2.) $x_i^{-1} t_i^{t_{(i,i)}^{-1}} \in T_{i-1}$ for all $i = 2, 3, \dots n$

then T is an overgroup of G of co-index,

$$|T : G| = \prod_{i=1}^{n} t_{i,i}^{-1}.$$

Proof. We note that once we have shown that $G \leq T$ we may use the above analysis to conclude that

$$|T : G| = \prod_{i=1}^{n} t_{i,i}^{-1}.$$

Thus it remains to show $G \leq T$, that is,

$$\langle x_n, x_{n-1}, \dots, x_1 \rangle \leq T.$$

So it suffices to show that $x_i \in T_i$ for every i. Hence for fixed i we note that (1) means that

$$t_i^{t_{(i,i)}^{-1}} = (x_i^{t_{i,i}} x_{i-1}^{t_{i,(i-1)}} \cdots x_1^{t_{i,1}})^{t_{(i,i)}^{-1}} = x_i \cdot (x_{i-1}^{\theta_{i-1}} x_{i-2}^{\theta_{i-2}} x_1^{\theta_1})$$

for some $\theta_i \in \mathbb{Q}$ Then condition (2) will ensure that

$$(x_{i-1}^{\theta_{i-1}} x_{i-2}^{\theta_{i-2}} x_1^{\theta_1}) \in T_{i-1}$$

so that $x_i \in T_i$. Hence we are done. $\square$

3.2 The Co-index Zeta Function of C_∞^d

We now wish to calculate the co-index zeta function of C_∞^d. We do this in two ways: using the above method and utilising the duality theorem of finite abelian groups. The second of these two proofs is more elegant but the first is included to illustrate this effective technique for use when the more elegant methods do not apply.

Theorem 3.5. *Let $G = C_\infty^d$ then*

$$\zeta_G^{up}(s) = \zeta_G(s) = \zeta(s)\zeta(s-1)\cdots\zeta(s-d+1).$$

Before this however we prove the following (easier) case:

Lemma 3.6. *Let $G = C_\infty^2$ then*

$$\zeta_G^{up}(s) = \zeta_G(s) = \zeta(s)\zeta(s-1).$$

Proof. Let $G = \langle x, y\rangle = C_\infty^2$. Then the upper central series for G is

$$1 \trianglelefteq G.$$

We refine this to

$$G_0 = 1 \trianglelefteq G_1 = \langle y\rangle \trianglelefteq G_2 = \langle x, y\rangle = G.$$

Next we form the Mal'cev completion for G. We observe that as $G \simeq \mathbb{Z}^2$ and that $(\mathbb{Z}^2)^{\mathbb{Q}} = \mathbb{Q}^2$ we have $G^{\mathbb{Q}} \simeq \mathbb{Q}^2$.

We wish to count finite index overgroups of T of G and so we seek valid bases of these overgroups. That is we wish to find $t_2 = x_2^{t_{2,2}} x_1^{t_{2,1}} \in G_2^{\mathbb{Q}} - G_1^{\mathbb{Q}}$ and $t_1 = x_1^{t_{1,1}} \in G_1^{\mathbb{Q}}$ such that:

1. t_2, t_1 is reduced
2. $T_2 = \{t_2^a t_1^b | a, b \in \mathbb{Z}\}$ and $T_1 = \{t_1^c | c \in \mathbb{Z}\}$ are groups, *i.e* $[t_2, t_1] \in T_1$

3. $t_{1,1} = \frac{1}{\alpha}$ and $t_{2,2} = \frac{1}{\beta}$ for $\alpha, \beta \in \mathbb{Z} - \{0\}$

4. $x_2^{-1} t_2^{\beta} \in T_1$.

As we are dealing with an abelian group condition (2) is trivial. Condition (3) tells us we must have

$$\begin{aligned} t_2 &= x_2^{\frac{1}{\beta}} \ x_1^{\gamma} \\ t_1 &= \qquad x_1^{\frac{1}{\alpha}} \end{aligned}$$

and then condition(4) ensures that

$$\begin{aligned} x_2^{-1} t_2^{\beta} &= x_2^{-1}(x_2^{\frac{1}{\beta}} x_1^{\gamma})^{\beta} \\ &= x_1^{\gamma\beta} \in T_1, \end{aligned}$$

in other words we must have that $\beta\gamma = \frac{\lambda}{\alpha}$ for some $\lambda \in \mathbb{Z}$.

It remains to consider when t_2, t_1 will be reduced. As we must have that $t_1 \sqsubset_{\mathbb{Q}} t_1^{-1}$ we need $\alpha \in \mathbb{N}$. Similarly $t_2 \sqsubset_{\mathbb{Q}} t_2^{-1}$ means that

$$t_{2,2} = \frac{1}{\beta} \prec_{\mathbb{Q}} -\frac{1}{\beta}$$

that is, we need $\beta \in \mathbb{N}$. Being reduced also means that

$$t_2 \sqsubset_{\mathbb{Q}} t_2 t_1^{\pm 1},$$

so we have

$$x_2^{\frac{1}{\beta}} x_1^{\gamma} \sqsubset_{\mathbb{Q}} x_2^{\frac{1}{\beta}} x_1^{\gamma \pm \frac{1}{\alpha}}.$$

In order for this to hold we must have

$$\gamma \prec_{\mathbb{Q}} \gamma \pm \frac{1}{\alpha}.$$

We break this into two cases $\gamma \leq 0$ and $\gamma > 0$. If $\gamma \leq 0$ we must have that

$$\gamma \prec_{\mathbb{Q}} \gamma - \frac{1}{\alpha} \tag{3.2}$$

$$\gamma \prec_{\mathbb{Q}} \gamma + \frac{1}{\alpha}. \tag{3.3}$$

As $\gamma - \frac{1}{\alpha}$ is more negative than γ it is clear (3.2) holds. For (3.3) we need

$$\gamma > \gamma + \frac{1}{\alpha}$$

which cannot happen.

So it must be the case that $\gamma > 0$ and then we consider when

$$\gamma \prec_{\mathbb{Q}} \gamma - \frac{1}{\alpha} \tag{3.4}$$
$$\gamma \prec_{\mathbb{Q}} \gamma + \frac{1}{\alpha}. \tag{3.5}$$

It is clear that (3.5) holds as we are only dealing with positive numbers. For (3.4) to hold we must have that

$$\gamma - \frac{1}{\alpha} \leq 0$$

so that

$$0 < \gamma \leq \frac{1}{\alpha}.$$

We know from above that

$$\beta\gamma = \frac{\lambda}{\alpha} \text{ for some } \lambda \in \mathbb{Z}$$

or equivalently $\alpha\beta\gamma \in \mathbb{Z}$. This allows us to write

$$\gamma = \frac{\lambda}{\alpha\beta}$$

for some $\lambda \in \mathbb{Z}$. Thus we have the condition

$$0 < \frac{\lambda}{\alpha\beta} \leq \frac{1}{\alpha}$$

and hence

$$0 < \lambda \leq \beta.$$

Thus for a given α and β we have β distinct choices for γ.

Therefore we have that

$$\begin{aligned} t_2 &= x_2^{\frac{1}{\beta}} \; x_1^{\gamma} \\ t_1 &= \qquad x_1^{\frac{1}{\alpha}} \end{aligned}$$

is a good basis if $\alpha, \beta \in \mathbb{N}$ and $\gamma = \frac{\lambda}{\alpha\beta}$ for $0 < \lambda \leq \beta$. Thus,

$$\begin{aligned}\zeta^{\mathrm{up}}_{C^2_\infty}(s) &= \sum_{\alpha,\beta\in\mathbb{N}} \alpha^{-s}\beta^{-s+1} \\ &= \zeta(s)\zeta(s-1).\end{aligned}$$

□

We now use a more elegant method to prove the theorem. Firstly we recall the duality theorem for finite abelian groups.

Theorem (Duality Theorem of Finite Abelian Groups). *Suppose that we have a finite abelian group G. Let $G^* = \{f | f : G \to \mathbb{C}-\{0\}, f$ a homomorphism$\}$. Now G^* has a natural group structure and and $G \simeq G^*$.*

Corollary 3.7. *If G is a finite abelian group then the number of subgroups of index m is the same as the number of subgroups of size m.*

Proof. Let G be a finite abelian group. Then, by the structure theorem for finite abelian groups we have

$$G \simeq C_{d_1} \times C_{d_2} \times \cdots C_{d_t}$$

where $1 < d_i | d_{i+1}$ for all $1 \leq i < t$ (for more details see, for example, [ST00, Theorem 2.4.2]). Now we have that

$$G^* = \mathrm{Hom}(G, \mathbb{C}^*)$$

and we know that $\mathrm{Hom}(G, \mathbb{C}^*)$ will respect $\times$ so that

$$(A \times B)^* \simeq (A^* \times B^*)$$

(where the isomorphism is natural). Thus it suffices to prove the theorem in the case where G is a cyclic group. Let $G = C_n$, and note that the operation of forming G^* inverts the lattice of G, thus proving the corollary. □

Proof of Theorem 3.5. As $C^d_\infty \simeq \mathbb{Z}^d$ we prove the result for $\mathbb{Z}^d$.

Let $A = \mathbb{Z}^d$ and $\mathcal{U} = A^{\mathbb{Q}} \simeq \mathbb{Q}^d$. We seek the number of overgroups B of A such that $A \leq_f B \leq \mathcal{U}$ and $|B : A| = m$ for some fixed $m \in \mathbb{N}$. Let

$$C = \frac{1}{m}A = \left\{ \left(\frac{a_1}{m}, \frac{a_2}{m}, \ldots, \frac{a_d}{m}\right) | a_i \in \mathbb{Z} \text{ for } i = 1, 2, \ldots, d \right\}$$

and observe that we have an epimorphism,

$$\begin{aligned}\theta : C &\rightarrow \oplus_{i=1}^{d}\mathbb{Z}/m\mathbb{Z}\\ (\frac{a_1}{m},\frac{a_2}{m},\dots,\frac{a_d}{m}) &\mapsto (a_1 \bmod m, a_2 \bmod m, \dots, a_d \bmod m).\end{aligned}$$

This map has kernel, $\operatorname{Ker}\theta = \{(\frac{ma_1}{m},\frac{ma_2}{m},\dots,\frac{ma_d}{m})\} = A$ so that

$$C/A \simeq \oplus_{i=1}^{d}\mathbb{Z}/m\mathbb{Z}.$$

If $D \leq C/A$ and $|D| = m$, then we have that $D = E/A$ for some unique E with $A \leq E \leq \mathcal{U}$. Thus we need to understand subgroups of C/A of order m and then we can count overgroups of A of co-index m. We now apply the duality theorem of finite abelian groups to C/A

This tells us that

$$(C/A) \simeq (C/A)^*.$$

In other words this allows us to count the number of subgroups of index m instead of the number of subgroup of order m. Thus we need only calculate the number of groups F such that $A \leq F \leq C$ and $|C : F| = m$. However if $|C : F| = m$ we have that $mC \leq F$ and $mC = A \leq F$ so that we need only consider $F \leq C$ such that $|C : F| = m$. As $C \simeq A$ (using the map $(\frac{a_1}{m},\frac{a_2}{m},\dots,\frac{a_d}{m}) \mapsto (\frac{a_1\cdot m}{m},\frac{a_2\cdot m}{m},\dots,\frac{a_d\cdot m}{m})$) the number of groups of co-index m to A is the same as the number of subgroups of A of index m so that

$$\zeta^{\mathrm{up}}_{C^d_\infty}(s) = \zeta_{C^d_\infty}(s) = \zeta(s)\zeta(s-1)\cdots\zeta(s-d+1)$$

as required. □

Thus, as we are writing $\alpha_{\zeta^{\mathrm{up}}_{C^d_\infty}(s)}$ as $\alpha^{\mathrm{up}}_{C^d_\infty}$ we have,

Corollary 3.8. $\alpha^{up}_{C^d_\infty} = \alpha_{C^d_\infty} = d$

3.3 The Co-index Zeta Function of F_2^2

We now seek the co-index zeta function for the free nilpotent group of class 2 on 2 generators, the so-called Heisenberg group. Recall that

$$H = F_2^2 = \langle x, y, z | [x, y] = z, [x, z] = [y, z] = 1 \rangle.$$

We know from [Smi83, Theorem 1.3] and our earlier calculations that

$$\zeta_H(s) = \frac{\zeta(s)\zeta(s-1)\zeta(2s-2)\zeta(2s-3)}{\zeta(3s-3)}$$

so that $\alpha_H = 2$.

Theorem 3.9. *Let*

$$H = \langle x, y, z | [x, y] = z, [x, z] = [y, z] = 1 \rangle$$

then

$$\zeta_H^{up}(s) = \zeta(s)\zeta(2s-1)\zeta(2s-2).$$

We note that it is somewhat surprising that the upwards formula is 'nicer' than the downwards formula. Of course 'nicer' is very subjective, and just relates to the appearance of the formula. We also note that both formulas contain the factor of $\zeta(s)\zeta(2s-2)$.

Proof. Let $\mathcal{U} = H^{\mathbb{Q}}$ be the Mal'cev completion of the Heisenberg group. We note that we can write every element in $\mathcal{U}$ as $x^a y^b z^c$ for some $a, b, c \in \mathbb{Q}$. We also observe that in $\mathcal{U}$ we have

$$\begin{aligned} (x^a y^b z^c) \cdot (x^d y^e z^f) &= x^{a+d} y^{b+e} z^{c+f-bd} \\ (x^a y^b z^c)^n &= x^{na} y^{nb} z^{nc - \binom{n}{2} ab} \\ (x^a y^b z^c)^{-1} &= x^{-a} y^{-b} z^{-c-ab}. \end{aligned}$$

We wish to count the number of overgroups T of H. The upper central series of H is

$$1 \trianglelefteq \langle z \rangle \trianglelefteq H$$

which we refine to

$$1 \trianglelefteq \langle z \rangle \trianglelefteq \langle y, z \rangle \trianglelefteq \langle x, y, z \rangle = H$$

and we label $\langle z \rangle = H_1$, $\langle y, z \rangle = H_2$ and $H = H_3$. We seek valid bases for overgroups of T. That is we wish to find a good basis t_3, t_2, t_1 such that

1. $t_3 \in H_3^{\mathbb{Q}} - H_2^{\mathbb{Q}}$, $t_2 \in H_2^{\mathbb{Q}} - H_1^{\mathbb{Q}}$, $t_1 \in H_1^{\mathbb{Q}} - \{1\}$
2. t_3, t_2, t_1 reduced
3. $T_1 = \langle t_1 \rangle$, $T_2 = \langle t_1, t_2 \rangle$, $T_3 = \langle t_1, t_2, t_3 \rangle$ are groups
4. $t_{i,i} = 1/\alpha_i$ for some $\alpha_i \in \mathbb{Z}$
5. $x^{-1} t_3^{\alpha_3} \in T_2$ and $y^{-1} t_2^{\alpha_2} \in T_1$

(this is just applying the conditions for a good basis). Hence let

$$\begin{array}{rcccc} t_3 & = & x^{\alpha^{-1}} & y^{\beta} & z^{\gamma} \\ t_2 & = & & y^{\delta^{-1}} & z^{\varepsilon} \\ t_1 & = & & & z^{\lambda^{-1}} \end{array}$$

so that we have fulfilled condition (1) and we note that to meet condition (4) we must pick $\alpha, \delta, \lambda \in \mathbb{Z}$. In fact $\alpha, \delta, \lambda \in \mathbb{N}$, as otherwise the basis would not be reduced. Clearly T_1 and T_2 are groups and if

$$[t_3, t_2] \in T_2$$

we will have T_3 being a group. Now,

$$\begin{aligned} [t_3, t_2] &= [x^{\alpha^{-1}} y^{\beta} z^{\gamma}, y^{\delta^{-1}} z^{\varepsilon}] \\ &= [x^{\alpha^{-1}} y^{\beta}, y^{\delta^{-1}}] \text{ as } z \text{ is central} \\ &= z^{(\alpha\delta)^{-1}}. \end{aligned}$$

Hence we need that

$$(\alpha\delta)^{-1} = l\lambda^{-1} \quad \text{or} \quad \lambda = l\alpha\delta$$

for some $l \in \mathbb{Z}$. For t_3, t_2, t_1 to be reduced we need t_3 to be such that

$$t_3 \sqsubset_{\mathbb{Q}} t_3 t_2^{\pm 1}$$

that is we need

$$\begin{aligned} \beta &\prec_{\mathbb{Q}} \beta - \delta^{-1} && (3.6) \\ \text{and } \beta &\prec_{\mathbb{Q}} \beta + \delta^{-1}. && (3.7) \end{aligned}$$

We consider two cases. Firstly if $\beta \leq 0$ we need (as $\delta^{-1} > 0$)

$$\beta \prec_{\mathbb{Q}} \beta + \delta^{-1}$$

i.e. as $\beta < 0$ we need $\beta + \delta^{-1} < \beta$ and this is clearly false. Thus we must have $\beta > 0$. Hence (3.7) is clearly true and we only need check that

$$\beta \prec_{\mathbb{Q}} \beta - \delta^{-1}$$

that is we must have $\beta - \delta^{-1} \leq 0$ and so we insist that

$$0 < \beta \leq \delta^{-1}.$$

Similar arguments will show that we will also need

$$0 < \gamma, \varepsilon \leq \lambda^{-1}$$

if t_3, t_2, t_1 is to be reduced. Finally we need to satisfy condition (5),

$$y^{-1}t_2^{\delta} \in T_1 \quad \text{and} \quad x^{-1}t_3^{\alpha} \in T_2.$$

Firstly we note that

$$y^{-1}t_2^{\delta} = y^{-1} \cdot yz^{\delta\varepsilon} \in T_1 = \langle z^{\lambda^{-1}} \rangle$$

if $\delta\varepsilon = \frac{a}{\lambda}$ for some $a \in \mathbb{Z}$. Thus we need $\lambda\delta\varepsilon \in \mathbb{Z}$ or equivalently we need $\delta = \frac{a}{\varepsilon\lambda}$ for some $a \in \mathbb{Z}$.

Similarly we have

$$x^{-1}(x^{\alpha^{-1}}y^{\beta}z^{\gamma})^{\alpha} = x^{-1} \cdot xy^{\alpha\beta}z^{\alpha\gamma - \binom{\alpha}{2}\alpha^{-1}\beta} \in T_2$$

if

$$y^{\alpha\beta}z^{\alpha\gamma - \binom{\alpha}{2}\alpha^{-1}\beta} = t_2^b t_3^c$$

so we need

$$\begin{aligned} \alpha\beta &= b\delta^{-1} \text{ and} && (3.8)\\ \alpha\gamma - \binom{\alpha}{2}\alpha^{-1}\beta &= b\varepsilon + c\lambda^{-1}. && (3.9) \end{aligned}$$

For (3.8) we need that $\alpha\beta\delta \in \mathbb{Z}$ so that we set $\beta = \frac{b}{\alpha\delta}$ for some $b \in \mathbb{Z}$. To meet the condition (3.9), as b is now fixed as above we need

$$\alpha\gamma - \binom{z\alpha}{2}\alpha^{-1}\beta - b\varepsilon - c\lambda^{-1} = 0$$

that is we need

$$\alpha\gamma\lambda - \frac{\alpha(\alpha-1)\beta\lambda}{2\alpha} - b\varepsilon\lambda \in \mathbb{Z}.$$

Thus t_3, t_2, t_1 is a good basis if it meets the above conditions. We now fix α, δ and λ (such that $\lambda = l\alpha\delta$ for some $l \in \mathbb{Z}$) and we ask how many good bases this choice will give us. As we have

$$\beta = \frac{b}{\alpha\delta} \text{ and } 0 < \beta \leq \delta^{-1}$$

we can pick $0 < b \leq \alpha$ and so we have α valid choices for β. Similarly as,

$$\varepsilon = \frac{a}{\delta\lambda} \text{ and } 0 < \varepsilon \leq \lambda^{-1}$$

we have δ choices for ε. Finally we need that

$$\alpha\gamma\lambda - \frac{\alpha(\alpha-1)\beta\lambda}{2\alpha} - b\varepsilon\lambda \in \mathbb{Z}$$

and $0 < \gamma \leq \lambda^{-1}$ so if we set

$$\alpha\gamma\lambda = i + \frac{\alpha(\alpha-1)\beta\lambda}{2\alpha} - b\varepsilon\lambda$$

for some $i \in \mathbb{Z}$ we see that

$$\gamma = \frac{1}{\alpha\lambda}\left(i + \frac{(\alpha-1)\beta\lambda}{2} - b\varepsilon\lambda\right).$$

As $0 < \gamma \leq \lambda^{-1}$ we need to know when

$$0 < \frac{1}{\alpha\lambda}\left(i + \frac{(\alpha-1)\beta\lambda}{2} - b\varepsilon\lambda\right) \leq \lambda^{-1}$$

that is when
$$0 < \left(i + \frac{(\alpha-1)\beta\lambda}{2} - b\varepsilon\lambda\right) \leq \alpha.$$

Hence we need to know $|(\mathbb{Z}+d)\cap(0,\alpha]|$ for $d = \frac{(\alpha-1)\beta\lambda}{2} - b\varepsilon\lambda$. Clearly the size of this set is α. Thus we have $\alpha^2\delta$ choices for good basis of index $\alpha\delta\lambda$, subject to the condition that $\lambda = l\alpha\delta$. for $l \in \mathbb{Z}$. So,

$$\zeta_H^{\mathrm{up}}(s) = \sum_{\substack{\alpha\in\mathbb{N}\ \delta\in\mathbb{N}\ \lambda\in\mathbb{N} \\ \lambda = l\alpha\delta \text{ for some } l\in\mathbb{Z}}} (\alpha\delta\lambda)^{-s}\alpha^2\delta$$

As we know we have an Euler product formula for this zeta function we now switch to considering this for a given prime $p \in \mathcal{P}$. Hence we consider

$$\zeta_H^{p,\mathrm{up}}(s) = \sum_{\substack{a=0 \\ a+d\leq l}}^{\infty}\sum_{d=0}^{\infty}\sum_{l=0}^{\infty} (p^a p^d p^l)^{-s} p^{2a} p^d$$

and we then have
$$\zeta_H^{\mathrm{up}}(s) = \prod_{p\in\mathcal{P}} \zeta_H^{p,\mathrm{up}}(s).$$

If we let $i = a+d$ so we have, $0 \leq a \leq i \leq l$ so that $d = i-a$ and we fix the condition that $a+d \leq l$. Thus we have

$$\begin{aligned}
\zeta_H^{p,\mathrm{up}}(s) &= \sum_{l=0}^{\infty}\sum_{i=0}^{l}\sum_{a=0}^{i}(p^a p^{i-a} p^l)^{-s} p^{2a} p^{i-a} \\
&= \sum_{l=0}^{\infty}\sum_{i=0}^{l}\sum_{a=0}^{i}(p^i p^l)^{-s} p^a p^i \\
&= \sum_{l=0}^{\infty}\sum_{i=0}^{l}\sum_{a=0}^{i} p^{-ls} p^{i(1-s)} p^a.
\end{aligned}$$

As we need infinite sums, we alter the above to our advantage by summing over all possible values for a and the subtracting those which are not valid, i.e $a > i$.

Hence we have,

$$\begin{aligned}
\zeta_H^{p,\mathrm{up}}(s) &= \sum_{l=0}^{\infty}\sum_{i=0}^{l}\sum_{a=0}^{i} p^{-ls}p^{i(1-s)}p^{a} \\
&= \sum_{l=0}^{\infty}\sum_{i=0}^{l}\sum_{a=0}^{\infty} p^{-ls}p^{i(1-s)}p^{a} - \sum_{l=0}^{\infty}\sum_{i=0}^{l}\sum_{a=i+1}^{\infty} p^{-ls}p^{i(1-s)}p^{a} \\
&= \sum_{l=0}^{\infty}\sum_{i=0}^{l}\sum_{a=0}^{\infty} p^{-ls}p^{i(1-s)}p^{a} - \sum_{l=0}^{\infty}\sum_{i=0}^{l}\sum_{\substack{b=0,\\ a=i+1+b}}^{\infty} p^{-ls}p^{i(1-s)}p^{i+1+b} \\
&= \sum_{l=0}^{\infty}\sum_{i=0}^{l}\sum_{a=0}^{\infty} p^{-ls}p^{i(1-s)}p^{a} - p\left(\sum_{l=0}^{\infty}\sum_{i=0}^{l}\sum_{b=0}^{\infty} p^{-ls}p^{i(2-s)}p^{b}\right) \qquad (3.10)
\end{aligned}$$

where we have set $a = i + 1 + b$ and sum over b for 0 to ∞. If we now consider the first term in the above and again we change the summation to sum over all i and subtract the invalid ones we have,

$$\begin{aligned}
\sum_{l=0}^{\infty}\sum_{i=0}^{l}\sum_{a=0}^{\infty} p^{-ls}p^{i(1-s)}p^{a} &= \sum_{l=0}^{\infty}\sum_{i=0}^{\infty}\sum_{a=0}^{\infty} p^{-ls}p^{i(1-s)}p^{a} - \sum_{l=0}^{\infty}\sum_{i=l+1}^{\infty}\sum_{a=0}^{\infty} p^{-ls}p^{i(1-s)}p^{a} \\
&= \sum_{l=0}^{\infty}\sum_{i=0}^{\infty}\sum_{a=0}^{\infty} p^{-ls}p^{i(1-s)}p^{a} \\
&\quad -\sum_{l=0}^{\infty}\sum_{j=0,i=l+1+j}^{\infty}\sum_{a=0}^{\infty} p^{-ls}p^{(l+1+j)(1-s)}p^{a} \\
&= \frac{\zeta^p(s)\zeta^p(s-1)}{(1-p)} - \sum_{l=0}^{\infty}\sum_{j=0}^{\infty}\sum_{a=0}^{\infty} p^{l(1-2s)}p^{j(1-s)}p^{(1-s)}p^{a}
\end{aligned}$$

where again we have put $i = l + 1 + j$. Hence we have

$$\sum_{l=0}^{\infty}\sum_{i=0}^{l}\sum_{a=0}^{\infty} p^{-ls}p^{i(1-s)}p^{a} = \frac{\zeta^p(s)\zeta^p(s-1)}{(1-p)} - \frac{p^{(1-s)}\zeta^p(s-1)\zeta^p(2s-1)}{(1-p)}.$$

Returning to the second term of (3.10) we see that

$$\begin{aligned}
&p\left(\sum_{l=0}^{\infty}\sum_{i=0}^{l}\sum_{b=0}^{\infty} p^{-ls}p^{i(2-s)}p^{b}\right) = \\
&\qquad p\left(\frac{\zeta^p(s-2)\zeta^p(s)}{(1-p)} - p^{(2-s)}\sum_{l=0}^{\infty}\sum_{j=0}^{\infty}\sum_{b=0}^{\infty} p^{l(2-2s)}p^{j(2-s)}p^{b}\right)
\end{aligned}$$

where again we have switched from summing $i = 0$ to l to summing over all i

and subtracting off the invalid values. Hence we now have, combining the above

$$\begin{aligned}
\zeta_H^{p,\mathrm{up}}(s) &= \frac{\zeta^p(s)\zeta^p(s-1)}{(1-p)} - \frac{p^{(1-s)}\zeta^p(s-1)\zeta^p(2s-1)}{(1-p)} \\
&\qquad -p\frac{\zeta^p(s-2)\zeta^p(s)}{(1-p)} + p^{3-s}\frac{\zeta^p(2s-2)\zeta^p(s-2)}{(1-p)} \\
&= \frac{\zeta^p(s)}{(1-p)}\left[\zeta^p(s-1) - p\zeta^p(s-2)\right] \\
&\qquad + \frac{p^{1-s}}{(1-p)}\left[p^2\zeta^p(s-2)\zeta^p(2s-2) - \zeta^p(s-1)\zeta^p(2s-1)\right]
\end{aligned} \tag{3.11}$$

If we consider

$$\begin{aligned}
\zeta^p(s-1) - p\zeta^p(s-2) &= \frac{1}{(1-p^{1-s})} - \frac{p}{(1-p^{2-s})} \\
&= \frac{1-p^{2-s}-p+p^{2-s}}{(1-p^{1-s})(1-p^{2-s})} \\
&= \frac{(1-p)}{(1-p^{1-s})(1-p^{2-s})} \\
&= (1-p)\zeta^p(s-1)\zeta^p(s-2),
\end{aligned}$$

we see that (3.11) becomes

$$\begin{aligned}
\zeta_H^{p,\mathrm{up}}(s) &= \frac{(1-p)\zeta^p(s)\zeta^p(s-1)\zeta^p(s-2)}{(1-p)} \\
&\qquad + \frac{p^{1-s}}{(1-p)}\left[p^2\zeta^p(s-2)\zeta^p(2s-2) - \zeta^p(s-1)\zeta^p(2s-1)\right] \\
&= \zeta^p(s)\zeta^p(s-1)\zeta^p(s-2) \\
&\qquad + \frac{p^{1-s}}{(1-p)}\left[p^2\zeta^p(s-2)\zeta^p(2s-2) - \zeta^p(s-1)\zeta^p(2s-1)\right].
\end{aligned} \tag{3.12}$$

If we now consider, $p^2\zeta^p(s-2)\zeta^p(2s-2) - \zeta^p(s-1)\zeta^p(2s-1)$ we see that this becomes

$$
\begin{aligned}
&= \frac{p^2}{(1-p^{2-s})(1-p^{2-2s})} - \frac{1}{(1-p^{1-s})(1-p^{1-2s})} \\
&= \frac{p^2(1-p^{1-s})(1-p^{1-2s}) - (1-p^{2-s})(1-p^{2-2s})}{(1-p^{2-s})(1-p^{2-2s})(1-p^{1-s})(1-p^{1-2s})} \\
&= \frac{p^2 - p^{3-s} - p^{3-2s} + p^{4-3s} - (1-p^{2-2s}-p^{2-s}+p^{4-3s})}{(1-p^{2-s})(1-p^{2-2s})(1-p^{1-s})(1-p^{1-2s})} \\
&= \frac{(1-p)(1-p^{1-s})(-1-p-p^{1-s})}{(1-p^{2-s})(1-p^{2-2s})(1-p^{1-s})(1-p^{1-2s})} \\
&= \frac{(1-p)(-1-p-p^{1-s})}{\zeta^p(s-1)(1-p^{2-s})(1-p^{2-2s})(1-p^{1-s})(1-p^{1-2s})} \\
&= \frac{(1-p)(-1-p-p^{1-s})}{\zeta^p(s-1)}\zeta^p(s-2)\zeta^p(2s-2)\zeta^p(s-1)\zeta^p(2s-1).
\end{aligned}
$$

So (3.12) becomes

$$
\begin{aligned}
\zeta_H^{p,\mathrm{up}}(s) &= \zeta^p(s)\zeta^p(s-1)\zeta^p(s-2) \\
&\qquad + \frac{p^{1-s}(1-p)(-1-p-p^{1-s})}{(1-p)}\zeta^p(s-2)\zeta^p(2s-2)\zeta^p(2s-1) \\
&= \zeta^p(s)\zeta^p(s-1)\zeta^p(s-2) \\
&\qquad -(1+p+p^{1-s})p^{1-s}\zeta^p(s-2)\zeta^p(2s-2)\zeta^p(2s-1) \\
&= \zeta^p(s-2)\left[\zeta^p(s)\zeta^p(s-1) - (1+p+p^{1-s})p^{1-s}\zeta^p(2s-2)\zeta^p(2s-1)\right] \\
&= \zeta^p(s-2)\left[\frac{1}{(1-p^{-s})(1-p^{1-s})} - \frac{p^{1-s}(1+p+p^{1-s})}{(1-p^{2-2s})(1-p^{1-2s})}\right] \\
&= \frac{\zeta^p(s-2)}{(1-p^{2-2s})(1-p^{1-2s})(1-p^{-s})(1-p^{1-s})}\cdot \\
&\qquad \left[(1-p^{2-2s}-p^{1-2s}+p^{3-4s})\right. \\
&\qquad \left. -p^{1-s}(1+p+p^{1-s})(1-p^{-s}-p^{1-s}+p^{1-2s})\right].
\end{aligned}
$$

Now we notice that

$$
\begin{aligned}
&p^{1-s}(1+p+p^{1-s})(1-p^{-s}-p^{1-s}+p^{1-2s}) = \\
&\qquad\qquad p^{1-s}+p^{2-s}-p^{3-2s}-p^{1-2s}-p^{2-2s}+p^{3-4s}.
\end{aligned}
$$

So that

$$
(1-p^{2-2s}-p^{1-2s}+p^{3-4s}) - p^{1-s}(1+p+p^{1-s})(1-p^{-s}-p^{1-s}+p^{1-2s})
$$

becomes

$$(1 - p^{1-s} - p^{2-s} + p^{3-2s}) = (1 - p^{1-s})(1 - p^{2-s}).$$

Hence we have that

$$\begin{aligned}
\zeta_H^{p,\mathrm{up}}(s) &= \zeta^p(s-2)\left[\frac{(1-p^{1-s})(1-p^{2-s})}{(1-p^{2-2s})(1-p^{1-2s})(1-p^{-s})(1-p^{1-s})}\right] \\
&= \zeta^p(s)\zeta^p(2s-1)\zeta^p(2s-2).
\end{aligned}$$

Then taking this as a product over all primes yields

$$\zeta_H^{\mathrm{up}}(s) = \zeta(s)\zeta(2s-1)\zeta(2s-2)$$

as required. □

Corollary 3.10.

$$\alpha_{F_2^2}^{up} = 3/2$$

We note that this abscissa is lower than that for the downwards zeta function of the Heisenberg group, which has $\alpha_H = 2$.

3.4 The Co-index Zeta Function of $\mathcal{H}$-groups

We now wish to calculate the co-index zeta function for other $\mathcal{T}$-groups of class 2 and Hirsch length 3. These groups G (sometimes called $\mathcal{H}$-groups) are classified up to isomorphism by the single numerical invariant $|Z(G) : \gamma_2(G)|$ and may be presented as

$$\langle x, y, z | [x, y] = z^t, [x, z] = [y, z] = 1\rangle$$

for some $t \in \mathbb{N}$. For convenience we will denote such a group by H_t. Clearly H_t has upper central series

$$1 \trianglelefteq \langle z\rangle \trianglelefteq H_t$$

as $H_t/Z(H) \simeq \mathbb{Z}^2$. We refine this to

$$1 \trianglelefteq G_1 \trianglelefteq G_2 \trianglelefteq G_3 = H_t$$

where $G_1 = Z(H_t)$ and $G_2 = \langle y, z\rangle$. So to count overgroups T of H_t in its Mal'cev completion $\mathcal{U} = H_t^{\mathbb{Q}}$, we need to consider possible good bases which can occur for such T. We pick a candidate possible basis t_3, t_2, t_1 with $t_i \in G_i^{\mathbb{Q}} - G_{i-1}^{\mathbb{Q}}$ as follows

$$\begin{array}{rcccc} t_3 & = & x^{\alpha} & y^{\beta} & z^{\gamma} \\ t_2 & = & & y^{\delta} & z^{\varepsilon} \\ t_1 & = & & & z^{\lambda}. \end{array}$$

Now to be a suitable good basis, t_3, t_2, t_1 must to satisfy

1. t_3, t_2, t_1 reduced
2. T_i is a group for $i = 1, 2, 3$, i.e, $[t_i, t_k] \in T_{i-1}$ for $k < i$
3. $\alpha^{-1}, \delta^{-1}, \lambda^{-1} \in \mathbb{Z}$
4. $x^{-1}t_3^{\alpha} \in T_2$ and $y^{-1}t_2^{\delta} \in T_1$.

Condition (3) says that $\alpha^{-1}, \delta^{-1}, \lambda^{-1} \in \mathbb{Z}$. Given this fact we relabel the exponents α, δ, and λ to be $\alpha^{-1}, \delta^{-1}, \lambda^{-1}$ as follows

$$\begin{array}{rcccc} t_3 & = & x^{\alpha^{-1}} & y^{\beta} & z^{\gamma} \\ t_2 & = & & y^{\delta^{-1}} & z^{\varepsilon} \\ t_1 & = & & & z^{\lambda^{-1}}. \end{array}$$

We now check condition (2). Firstly note that $[t_3, t_1] = [t_2, t_1] = 1$ as z is central. We also observe that

$$[t_3, t_2] = t_3^{-1}t_2^{-1}t_3t_2 = z^{-t\alpha^{-1}\beta}.$$

If we are to have

$$z^{-t\alpha^{-1}\delta^{-1}} \in T_1 = \langle z^{\lambda^{-1}} \rangle$$

we require that $t\alpha^{-1}\delta^{-1} = a\lambda^{-1}$ for some $a \in \mathbb{Z}$ (or $t\lambda = a\alpha\delta$).

Thus it remains to check conditions (1) and (4). We show (1) first, i.e that t_3, t_2, t_1 are reduced. Thus for $i = 1, 2, 3$ we check

a. $t_i \sqsubset_{\mathbb{Q}} t_i^{-1}$

b. $t_i \sqsubset_{\mathbb{Q}} t_i t_j^a$ for $j < i$ and $a \in \{1, -1\}$.

To meet condition (a) it is sufficient to have $\alpha, \delta, \lambda \in \mathbb{N}$. For condition (b) we note that,

$$\begin{array}{rcl} t_3t_2 & = & x^{\alpha^{-1}} \quad y^{\beta+\delta^{-1}} \quad z^{\gamma+\varepsilon} \\ t_3t_2^{-1} & = & x^{\alpha^{-1}} \quad y^{\beta-\delta^{-1}} \quad z^{\gamma-\varepsilon} \\ t_2t_1 & = & y^{\delta^{-1}} \quad z^{\varepsilon+\lambda^{-1}} \\ t_2t_1^{-1} & = & y^{\delta^{-1}} \quad z^{\varepsilon-\lambda^{-1}}. \end{array}$$

So to have $t_3 \sqsubset_{\mathbb{Q}} t_3t_2$ and $t_3 \sqsubset_{\mathbb{Q}} t_3t_2^{-1}$ we need

$$\begin{array}{rcl} \beta & \prec_{\mathbb{Q}} & \beta + \delta^{-1} \text{ and} \\ \beta & \prec_{\mathbb{Q}} & \beta - \delta^{-1}. \end{array}$$

We consider the two cases when $\beta \leq 0$ and $\beta > 0$. Firstly if $\beta \leq 0$, then we must have that both

$$\begin{array}{rcl} \beta + \delta^{-1} & < & \beta \text{ and} \\ \beta - \delta^{-1} & < & \beta \end{array}$$

and clearly this cannot happen. Thus we must have $\beta > 0$. In this case we need

- $\beta + \delta^{-1} < \beta$, and
- either
  - $\beta < \beta - \delta^{-1}$ if $\beta - \delta^{-1} > 0$, or
  - $\beta - \delta^{-1} < \beta$ if $\beta - \delta^{-1} \leq 0$.

Clearly $\beta \not< \beta - \delta^{-1}$ so we must have $\beta - \delta^{-1} \leq 0$. We deduce that

$$0 < \beta \leq \delta^{-1}.$$

Similar arguments allow us to show that t_3, t_2, t_1 are reduced when

$$\alpha, \delta, \lambda \in \mathbb{N}, \quad 0 < \beta \leq \delta^{-1}, \quad 0 < \gamma, \varepsilon \leq \lambda^{-1}.$$

Thus is remains to determine exactly when condition (4) holds. Thus we need to ensure that both

$$\begin{array}{rcl} x^{-1}t_3^{\alpha} & \in & T_2 \text{ and} \\ y^{-1}t_2^{\delta} & \in & T_1 \end{array}$$

(as $z^{-1}t_3^{\lambda^{-1}} = 1$). If $y^{-1}t_2^{\delta} \in T_1$ we need that

$$y^{-1} \cdot (y^{\delta^{-1}} z^{\varepsilon})^{\delta} \in T_1,$$

that is $z^{\varepsilon\delta} = (z^{\lambda^{-1}})^a$. Thus it must be the case that $\varepsilon\delta = a\lambda^{-1}$ or $\varepsilon\delta\lambda \in \mathbb{Z}$.

Finally we check when $x^{-1}t_3^{\alpha} \in T_2$. Now,

$$\begin{aligned} x^{-1}t_3^{\alpha} &= x^{-1} \cdot (xy^{\alpha\beta} z^{\alpha\gamma - t\binom{\alpha}{2}\alpha^{-1}\beta}) \\ &= y^{\alpha\beta} z^{\alpha\gamma - t\binom{\alpha}{2}\alpha^{-1}\beta}. \end{aligned}$$

This will happen if we can find an appropriate $a \in \mathbb{Z}$ such that

$$\left(y^{\alpha\beta} z^{\alpha\gamma - t\binom{\alpha}{2}\alpha^{-1}\beta}\right) \cdot t_2^{a} \in T_1.$$

Such an a exists if we can write $a\delta^{-1} = -\alpha\beta$, that is if $\alpha\beta\delta \in \mathbb{Z}$. Thus we need

$$z^{\alpha\gamma - t\binom{\alpha}{2}\alpha^{-1}\beta} \cdot z^{\varepsilon\delta\alpha\beta} \in T_1$$

which means that we require that

$$\alpha\gamma - t\tbinom{\alpha}{2}\alpha^{-1}\beta - \varepsilon\delta\alpha\beta = c\lambda^{-1}$$

for some $c \in \mathbb{Z}$. Therefore we need,

$$\alpha\gamma\lambda - t\tbinom{\alpha}{2}\alpha^{-1}\beta\lambda - \varepsilon\delta\alpha\beta\lambda = c \in \mathbb{Z}.$$

We know that $0 < \gamma \leq \lambda^{-1}$, and so we reorganise the above,

$$\gamma = \frac{1}{\alpha\lambda}\left(c + t\tbinom{\alpha}{2}\beta + \varepsilon\delta\beta\right)$$

which enables us to count the valid choices for γ.

Thus for t_3, t_2, t_1 to be a good basis for an overgroup of co-index $\alpha\delta\lambda$, we need

$$\begin{aligned} &t\lambda = l\alpha\beta, \text{ for some } l \in \mathbb{Z} \\ &0 < \beta \leq \delta^{-1}, \quad \alpha\beta\delta \in \mathbb{Z} \\ &0 < \varepsilon \leq \lambda^{-1}, \quad \varepsilon\delta\lambda \in \mathbb{Z} \\ &0 < \gamma \leq \lambda^{-1}, \quad \gamma = \tfrac{1}{\alpha\lambda}\left(c + t\tbinom{\alpha}{2}\beta + \varepsilon\delta\beta\right) \text{ for some } c \in \mathbb{Z}. \end{aligned}$$

We now wish to count the choices we can make for β, γ and ε with α, δ and λ fixed. For β, as we must have $\alpha\beta\delta \in \mathbb{Z}$ so

$$\beta = \frac{a}{\alpha\delta}$$

for some $a \in \mathbb{Z}$ and as $0 < \beta \leq \delta^{-1}$. This means that $0 < a \leq \alpha$ so that we have α choices for β. Similarly for ε we need

$$\varepsilon = \frac{b}{\delta\lambda} \text{ for } b \in \mathbb{Z} \text{ and } 0 < \varepsilon \leq \lambda^{-1},$$

so that we have δ choices of ε. Finally as

$$\gamma = \frac{1}{\alpha\lambda}\left(c + t\tbinom{\alpha}{2}\beta + \varepsilon\delta\beta\right)$$

for some $c \in \mathbb{Z}$ we are interested in the number of c such that

$$0 < c + t\tbinom{\alpha}{2}\beta + \varepsilon\delta\beta \leq \alpha$$

This is just $|(\mathbb{Z}+d) \cap [0,\alpha)|$ where $d = c + t\binom{\alpha}{2}\beta + \varepsilon\delta\beta$. Clearly $|(\mathbb{Z}+d) \cap [0,\alpha)| = \alpha$. So for fixed α, δ, λ we have $\alpha^2\delta$ choices of overgroups T of this co-index. Thus the upwards zeta function for H_t is,

$$\zeta^{\mathrm{up}}_{H_t}(s) = \sum_{\substack{\alpha\in\mathbb{N} \\ t\lambda = l\alpha\beta, \text{ for some } l \in \mathbb{Z}}} \sum_{\delta\in\mathbb{N}} \sum_{\lambda\in\mathbb{N}} (\alpha\delta\lambda)^{-s}\alpha^2\delta$$

As this does not seem to lend itself to simplification, and we have an Euler product formula for the co-index zeta function of $\mathcal{T}$-groups, we fix a prime p and try to calculate $\zeta^{\mathrm{up},p}_{H_t}(s)$. Let the p-part of t be p^θ, the p-part of α be p^a, the p-part of δ be p^b and the p-part of λ be p^c, then this becomes,

$$\zeta^{\mathrm{up},p}_{H_t}(s) = \sum_{\substack{c=0 \\ 0 \leq a+b \leq c+\theta}}^{\infty} \sum_{b=0}^{\infty} \sum_{a=0}^{\infty} (p^a p^b p^c)^{-s} p^{2a} p^b$$

If we now let $i = a + b$ (so that $b = i - a$) we rewrite this sum as

$$\zeta^{\mathrm{up},p}_{H_t}(s) = \sum_{c=0}^{\infty} \sum_{i=0}^{c+\theta} \sum_{a=0}^{i} (p^a p^{i-a} p^c)^{-s} p^{2a} p^{i-a} = \sum_{c=0}^{\infty} \sum_{i=0}^{c+\theta} \sum_{a=0}^{i} (p^i p^c)^{-s} p^a p^i$$

If we now switch from summing $a = 1$ to i to summing over all a and subtracting those $a > i$ we get

$$\begin{aligned}\zeta_{H_t}^{\mathrm{up}}(s) &= \sum_{c=0}^{\infty}\sum_{i=0}^{c+\theta}\sum_{a=0}^{\infty}(p^i p^c)^{-s}p^a p^i - \sum_{c=0}^{\infty}\sum_{i=0}^{c+\theta}\sum_{a=i+1}^{\infty}(p^i p^c)^{-s}p^a p^i \\ &= \sum_{c=0}^{\infty}\sum_{i=0}^{c+\theta}\sum_{a=0}^{\infty}(p^i p^c)^{-s}p^a p^i - \sum_{c=0}^{\infty}\sum_{i=0}^{c+\theta}\sum_{e=0}^{\infty}(p^i p^c)^{-s}p^{(i+1+e)}p^i\end{aligned}$$

where we have put $a = e + i + 1$. If we now use a similar trick with i in the first term we see that

$$\begin{aligned}\zeta_{H_t}^{\mathrm{up}}(s) &= \sum_{c=0}^{\infty}\sum_{i=0}^{\infty}\sum_{a=0}^{\infty}(p^i p^c)^{-s}p^a p^i - \sum_{c=0}^{\infty}\sum_{i=c+\theta+1}^{\infty}\sum_{a=0}^{\infty}(p^i p^c)^{-s}p^a p^i \\ &\qquad - \sum_{c=0}^{\infty}\sum_{i=0}^{c+\theta}\sum_{e=0}^{\infty}(p^i p^c)^{-s}p^{2i}p^e p \\ &= \frac{\zeta^p(s)\zeta^p(s-1)}{(1-p)} - \sum_{c=0}^{\infty}\sum_{j=0}^{\infty}\sum_{a=0}^{\infty}(p^{j+c+\theta+1}p^c)^{-s}p^a p^{j+c+\theta+1} \\ &\qquad - \sum_{c=0}^{\infty}\sum_{i=0}^{c+\theta}\sum_{e=0}^{\infty}(p^i p^c)^{-s}p^{2i}p^e p \\ &= \frac{\zeta^p(s)\zeta^p(s-1)}{(1-p)} - \sum_{c=0}^{\infty}\sum_{j=0}^{\infty}\sum_{a=0}^{\infty}p^{\theta(1-s)}p^{(1-s)}(p^j p^{2c})^{-s}p^a p^j p^c \\ &\qquad - \sum_{c=0}^{\infty}\sum_{i=0}^{c+\theta}\sum_{e=0}^{\infty}(p^i p^c)^{-s}p^{2i}p^e p\end{aligned}$$

where we have set $i = c + \theta + 1 + j$. If we now do the same trick on the last term of the above sum and tidy up we get,

$$\begin{aligned}
&= \frac{\zeta^p(s)\zeta^p(s-1)}{(1-p)} - \frac{p^{\theta(1-s)}p^{(1-s)}\zeta^p(s-1)\zeta^p(2s-1)}{(1-p)} \\
&\qquad -\sum_{c=0}^{\infty}\sum_{i=0}^{\infty}\sum_{e=0}^{\infty}(p^i p^c)^{-s}p^{2i}p^e p + \sum_{c=0}^{\infty}\sum_{i=c+\theta+1}^{\infty}\sum_{e=0}^{\infty}(p^i p^c)^{-s}p^{2i}p^e p \\
&= \frac{\zeta^p(s)\zeta^p(s-1)}{(1-p)} - \frac{p^{\theta(1-s)}p^{(1-s)}\zeta^p(s-1)\zeta^p(2s-1)}{(1-p)} - \\
&\qquad \frac{p\zeta^p(s)\zeta^p(s-2)}{(1-p)} + \sum_{c=0}^{\infty}\sum_{j=0}^{\infty}\sum_{e=0}^{\infty}(p^{j+c+\theta+1}p^c)^{-s}p^{2(j+c+\theta+1)}p^e p \\
&= \frac{\zeta^p(s)\zeta^p(s-1)}{(1-p)} - \frac{p\zeta^p(s)\zeta^p(s-2)}{(1-p)} - \\
&\qquad \frac{p^{\theta(1-s)}p^{(1-s)}\zeta^p(s-1)\zeta^p(2s-1)}{(1-p)} + \\
&\qquad \frac{p^{\theta(2-s)}p^{(3-s)}\zeta^p(2s-2)\zeta^p(s-2)}{(1-p)}.
\end{aligned}$$

We now begin to simplify the above by combining like terms

$$\begin{aligned}
&= \frac{\zeta^p(s)}{(1-p)}\left[\frac{1}{(1-p^{1-s})} - \frac{p}{(1-p^{2-s})}\right] + \\
&\qquad \frac{p^{\theta(2-s)}p^{(3-s)}\zeta^p(2s-2)\zeta^p(s-2)}{(1-p)} - \frac{p^{\theta(1-s)}p^{(1-s)}\zeta^p(s-1)\zeta^p(2s-1)}{(1-p)} \\
&= \frac{\zeta^p(s)\zeta^p(s-1)\zeta^p(s-2)}{(1-p)}\left[(1-p^{2-s}-p+p^{2-s}\right] + \\
&\qquad \frac{p^{\theta(2-s)}p^{(3-s)}\zeta^p(2s-2)\zeta^p(s-2)}{(1-p)} - \frac{p^{\theta(1-s)}p^{(1-s)}\zeta^p(s-1)\zeta^p(2s-1)}{(1-p)} \\
&= \zeta^p(s)\zeta^p(s-1)\zeta^p(s-2) + \\
&\qquad \frac{p^{\theta(2-s)}p^{(3-s)}\zeta^p(2s-2)\zeta^p(s-2)}{(1-p)} - \frac{p^{\theta(1-s)}p^{(1-s)}\zeta^p(s-1)\zeta^p(2s-1)}{(1-p)} \\
&= (1-p^{\theta(2-s)})\zeta^p(s)\zeta^p(s-1)\zeta^p(s-2) + \\
&\qquad p^{\theta(2-s)}\zeta^p(2s-1)\zeta^p(2s-2) + \\
&\qquad \frac{(p^\theta-1)p^{\theta(1-s)}p^{(1-s)}\zeta^p(s-1)\zeta^p(2s-1)}{(1-p)}.
\end{aligned}$$

This last step is computed using maple. We then have shown

Theorem 3.11. *Let $H_t \in \mathcal{H}$ then,*

$$\zeta_{H_t}^{up,p}(s) = (1 - p^{\theta(2-s)})\zeta_{\mathbb{Z}^3}^{up,p}(s) + p^{\theta(2-s)}\zeta_H^{up,p}(s) + \frac{(p^\theta - 1)p^{\theta(1-s)}p^{(1-s)}\zeta^p(s-1)\zeta^p(2s-1)}{(1-p)}.$$

Now we have

$$\zeta_{\mathbb{Z}^3}^{p,\mathrm{up}}(s) = \frac{1}{(1-p^{-s})(1-p^{1-s})(1-p^{2-s})} \tag{3.13}$$

and

$$\zeta_H^{p,\mathrm{up}}(s) = \frac{1}{(1-p^{-s})(1-p^{1-2s})(1-p^{2-2s})}. \tag{3.14}$$

From (3.14) we see that $p^{\theta(2-s)}\zeta_H^{p,\mathrm{up}}(s)$ is a non-zero analytic function for s such that $\mathrm{Re}\, s > 3/2$. From (3.13) we have

$$(1 - p^{\theta(2-s)})\zeta_{\mathbb{Z}^3}^{p,\mathrm{up}}(s) = \frac{1 - p^{\theta(2-s)}}{(1-p^{2-s})}\zeta_{\mathbb{Z}^2}^{p,\mathrm{up}}$$

and then

$$\frac{1 - p^{\theta(2-s)}}{(1-p^{2-s})} = 1 + p^{2-s} + \cdots + p^{(1-\theta)(2-s)}$$

is an analytic function for all s. Moreover $\zeta_{\mathbb{Z}^2}^{p,\mathrm{up}}(s)$ is an analytic for all s with $\mathrm{Re}\, s > 1$. Finally as we have that

$$\frac{(p^\theta - 1)}{(1-p)}p^{(1-s)}\zeta^p(s-1)\zeta^p(2s-1)$$

is an analytic function for all s with $\mathrm{Re}\, s > 2$. Thus we have shown that

$$\zeta_{H_t}^{\mathrm{up}}(s) = \prod_{p \in \mathcal{P}} \zeta_{H_t}^{p,\mathrm{up}}$$

is analytic for all s with $\mathrm{Re}\, s > 2$. Hence we have proved

Corollary 3.12. *If $G \in \mathcal{H}$ then $\alpha^{up} = 3/2$.*

3.5 Co-index Zeta Functions of Non-$\mathcal{T}$-groups

We now attempt to calculate the co-index zeta function for some other groups. We consider the plane crystallographic groups. This set of seventeen groups have

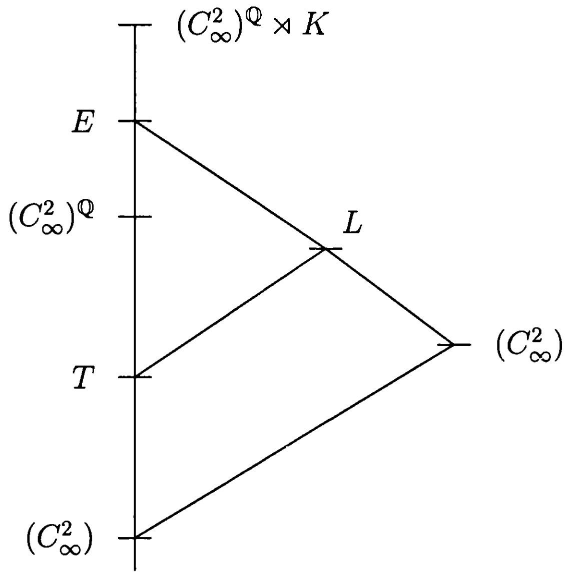


Figure 3.1: The Crystallographic case

had their zeta functions explicitly calculated in [McD97] (and see also [dSMS99]).

one of these groups (bar one which is C_∞^2) is nilpotent and all (again bar C_∞^2) have torsion. Thus we cannot use the Mal'cev completion, for it does not exist. However some of these groups can be written as split extensions, $C_\infty^2 \rtimes K$, for some appropriate finite group K. We begin by giving a method to calculate the co-index zeta function within the universe $(C_\infty^2)^{\mathbb{Q}} \rtimes K$. This method closely mimics the method given for the subgroup case, see [McD97]. This however may not be the 'best' universe to choose, as to some extent we are ignoring growth in the "K direction."

Let $C_\infty^2 = \langle x, y \rangle$ and form the groups $G = C_\infty^2 \rtimes K$ and $\mathcal{U} = (C_\infty^2)^{\mathbb{Q}} \rtimes K$. We then seek groups L such that $G \leq_f L \leq \mathcal{U}$.

Suppose that $(C_\infty^2)^{\mathbb{Q}} \leq_f E \leq \mathcal{U}$. To enable us to count all finite co-index groups of G we firstly seek groups L such that

- $L(C_\infty^2)^{\mathbb{Q}} = E$
- $L \cap (C_\infty^2)^{\mathbb{Q}} = T \geq_f C_\infty^2$
- $|L : G| < \infty$

and then sum over all such E to calculate the upwards zeta function.

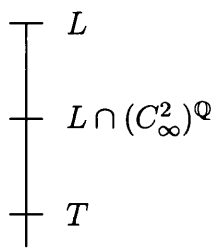
We already know how to describe $T \geq_f C_\infty^2$ from previous analysis of this case. For groups L as above to exists a necessary condition is that $T \trianglelefteq E$. We determine this by choosing elements X of E which generate E modulo $(C_\infty^2)^{\mathbb{Q}}$. We can then determine whether or not $T \trianglelefteq E$ by conjugation of a basis of T by elements of X and testing for membership of T.

We now assume that $T \trianglelefteq E$. If $L \leq E$ such that $L \cap (C_\infty^2)^{\mathbb{Q}} = T$ and $L(C_\infty^2)^{\mathbb{Q}} = E$ then there is a natural isomorphism between

$$E/(C_\infty^2)^{\mathbb{Q}} \text{ and } L/T$$

which will send preferred generators to preferred generators.

This process is reversible. Suppose we have elements $Y \subset E$ in a distinguished correspondence with X such that Y generates E modulo T. Suppose further that corresponding elements coincide modulo $(C_\infty^2)^{\mathbb{Q}}$. Find a set of relations R for $E/(C_\infty^2)^{\mathbb{Q}}$ on the generating set X. Let $L = \langle Y, T \rangle$ and suppose that Y satisfies R. We can apply Von Dyck's theorem to give us an epimorphism,



$$\beta : L/T \to E/(C_\infty^2)^{\mathbb{Q}}.$$

We know that $E/(C_\infty^2)^{\mathbb{Q}}$ is a finite group and we also have that

$$E/(C_\infty^2)^{\mathbb{Q}} \simeq L/(L \cap (C_\infty^2)^{\mathbb{Q}}).$$

The first isomorphism theorem tells us that

$$L/T/(\text{Ker } \beta) \simeq L/(L \cap (C_\infty^2)^{\mathbb{Q}})$$

thus we have that Ker $\beta = 1$ so that $L \cap (C_\infty^2)^{\mathbb{Q}} = T$ and hence $E = L(C_\infty^2)^{\mathbb{Q}}$.

We now use this method to attempt to calculate the upwards zeta function for the crystallographic group **pm** which can be presented

$$\mathbf{pm} = \langle x, y, m | [x, y], m^2, [x, m], y^m = y^{-1} \rangle$$

or written as $\mathbf{pm} = C_\infty^2 \rtimes C_2$. Thus we have that $\mathcal{U} = (C_\infty^2)^{\mathbb{Q}} \rtimes C_2$. Firstly we count groups $G \leq_f L \leq \mathcal{U}$ such that $L(C_\infty^2)^{\mathbb{Q}} = \mathcal{U}$ (that is we take $E = \mathcal{U}$). We note that if $C_\infty^2 \leq T \leq (C_\infty^2)^{\mathbb{Q}}$ we have that

$$T = \begin{pmatrix} x^{a^{-1}} & y^b \\ & y^{c^{-1}} \end{pmatrix}$$

with $a, c \in \mathbb{N}$ and $b = \frac{\varepsilon}{ac}$ with $0 < \varepsilon \leq a$. We pick a distinguished generator m of $\mathcal{U}/(C_\infty^2)^{\mathbb{Q}}$ and observe that if $T \trianglelefteq \mathcal{U}$ we must have,

$$\begin{aligned} (x^{a^{-1}} y^b)^m &\in T \\ (y^{c^{-1}})^m &\in T. \end{aligned}$$

We also need $z = (mx^d y^e)$ with $d, e \in \mathbb{Q}$, such that $z^2 \in T$. So overall we insist that,

$$\begin{aligned} (x^{a^{-1}} y^b)^m &= x^{a^{-1}} y^{-b} \in T && (3.15) \\ (y^{c^{-1}})^m &= y^{-c^{-1}} \in T && (3.16) \\ (mx^d y^e)^2 &= x^{2d} \in T. && (3.17) \end{aligned}$$

Observing that (3.16) is trivial and as T is generated by $x^{a^{-1}} y^b$ and $y^{c^{-1}}$ we must have that

$$\begin{aligned} x^{a^{-1}} y^{-b} &= (x^{a^{-1}} y^b)^\alpha (y^{c^{-1}})^\beta \\ x^{2d} &= (x^{a^{-1}} y^b)^\gamma (y^{c^{-1}})^\delta \end{aligned}$$

for $\alpha, \beta, \gamma, \delta \in \mathbb{Z}$. Observe that to avoid over counting we must have $0 < d \leq a^{-1}$ and $0 < e \leq c^{-1}$. Hence we need to show the following equations hold

$$\begin{aligned} a^{-1} &= \alpha \cdot a^{-1} && (3.18) \\ -b &= \alpha \cdot b + \beta \cdot c^{-1} && (3.19) \\ 2d &= \gamma \cdot a^{-1} && (3.20) \\ 0 &= \gamma \cdot b + \delta \cdot c^{-1}. && (3.21) \end{aligned}$$

Now (3.18) implies that $\alpha = 1$ which forces

$$\beta = -2bc = \frac{-2\varepsilon}{a}$$

so that $\varepsilon = a$ or $\varepsilon = \frac{a}{2}$. Now (3.20) tells us that

$$\gamma = 2da$$

and if $2da \in \mathbb{Z}$ we must have $d = \frac{1}{2a}$ or $d = \frac{1}{a}$. It remains to consider (3.21),

$$\delta = -\gamma bc.$$

As $\delta \in \mathbb{Z}$ and $\gamma = 1, 2$ this forces $\varepsilon = a$ if $\gamma = 1$ or we can have $\varepsilon = a$ or $\varepsilon = \frac{a}{2}$ if $\gamma = 2$.

Thus if $d = \frac{1}{2a}$ we must have $\varepsilon = a$ and if $d = \frac{1}{a}$ we can have both of $\varepsilon = a$ or $\varepsilon = \frac{a}{2}$.

However we still only have the condition that $0 < e \leq c^{-1}$ leaving us with a countable number of choices for e. Thus we cannot write down the co-index zeta function for this group in $\mathcal{U}$. The other plane crystallographic groups also exhibit this phenomenon.

Chapter 4

Normal and Co-normal Zeta functions

We now introduce a new zeta (and co-index zeta) function, the Normal Zeta function. The idea for this stems from the fact that zeta functions themselves may not encapsulate the information about a group that is of interest in a particular context. We will in §4.1 introduce the normal zeta function and give some results about it, before proceeding to the co-normal zeta function case.

4.1 Normal Zeta functions

We first make a definition.

Definition ([Smi83, Definition 1.6]). *Let G be a finitely generated group. The* normal zeta function *of G is*

$$\zeta_G^{\triangleleft}(s) = \sum_{H \trianglelefteq_f G} |G : H|^{-s} = a_n n^{-s}$$

where $a_n = a_n(G) = \{H | H \trianglelefteq G, |G : H| = n\}$*.*

As with the zeta function, we may relax the condition that G is finitely generated if for some other reason we know that each a_n is finite. We can also show the following, as noted in [Smi83],

Lemma 4.1. *Let $G \in \mathcal{T}$, then we have*

$$\zeta_G^{\triangleleft}(s) = \prod_{p \in \mathcal{P}} \zeta_G^{p,\triangleleft}(s)$$

where

$$\zeta_G^{p,\triangleleft}(s) = \sum_{\substack{H \trianglelefteq_f G \\ |G:H| \text{ a } p\text{-power}}} |G:H|^{-s}.$$

Proof. Let $H \trianglelefteq_f G$. We seek a collection of groups

$$\mathcal{K} = \{K_p | K_p \trianglelefteq G, |G:K_p| = p^{\lambda_p}, p \in \mathcal{P}\}$$

such that $\cap_{K_p \in \mathcal{K}} K_p = H$ and $K_p = G$ for all but finitely many primes $p \in \mathcal{P}$. To do this we use some of the theory of isolators. Let $p \in \mathcal{P}$ and define $\pi = \mathcal{P} - \{p\}$. Set

$$K_p = I_G^{\pi}(H) = \{g \in G | m \in \mathbb{N}, m \text{ a } \pi\text{-number }, g^m \in H\}$$

that is K_p is the π-isolator of H in G. We firstly note that $K_p \trianglelefteq G$ since if $k \in K_p$ then

$$(g^{-1}kg)^m = g^{-1}k^m g$$

for m a π-number such that $k^m \in H$. Thus we have $g^{-1}k^m g \in H$ as $H \trianglelefteq G$ and so by definition of K_p we have $(g^{-1}kg) \in K_p$.

We recall the fact that if $N \trianglelefteq_f G$ then N is π-isolated if and only if G/N is π-free. Thus we have that

$$|G:K_p| = p^{\lambda_p}$$

for some $\lambda_p \in \mathbb{N}$ as K_p is π-isolated. Hence, we also know that if $p \nmid |G:H|$ then $K_p = G$. Thus we have formed the desired collection $\mathcal{K}$ and so have shown the existence of the desired Euler product. $\square$

We now find the normal zeta function of the groups for which we have zeta functions. First of all (and easiest), as C_∞^d is abelian, we have

Proposition 4.2. *Let $G = C_\infty^d$, then*

$$\zeta_G^{\triangleleft}(s) = \zeta_G(s) = \zeta(s)\zeta(s-1)\cdots\zeta(s-d+1).$$

Corollary 4.3. *Let $G = C_\infty^d$ then $\alpha_G^{\triangleleft} = d$.*

(Where $\alpha_G^{\triangleleft}$ is the abscissa of convergence of $\zeta_G^{\triangleleft}(s)$.)

Next we consider $\mathcal{H}$-groups. We recall that if $H \trianglelefteq G$ then it must be the case that $[G, H] \leq H$. As we will be dealing with finitely generated groups, we can test this condition by considering $[g, h]$ for all generators g of G and h of H. Thus given a group $G \in \mathcal{H}$, with $G = \langle x, y, z | [x, y] = z^t, [x, z] = [y, z] = 1\rangle$ and a subgroup H given by a good basis

$$\begin{array}{ccc} x^{\alpha} & y^{\beta} & z^{\gamma} \\ & y^{\delta} & z^{\varepsilon} \\ & & z^{\lambda} \end{array}$$

we must test when

$$[x, x^{\alpha}y^{\beta}z^{\gamma}] \in H, [x, y^{\delta}z^{\varepsilon}] \in H, [y, x^{\alpha}y^{\beta}z^{\gamma}] \in H \text{ and } [y, y^{\delta}z^{\varepsilon}] \in H$$

(as z is central). Thus using conditions that the above will give and combining them with the good basis proof of the zeta function of G we can calculate the following results first given in [Smi83].

Theorem 4.4. *[Smi83, Theorem 1.5] Let $G = F_2^2$ then we have*

$$\zeta_G^{\triangleleft}(s) = \zeta(s)\zeta(s-1)\zeta(3s-2).$$

Corollary 4.5. *Let $G = F_2^2$ then $\alpha_G^{\triangleleft} = 2$.*

We can also do this for the other $\mathcal{H}$-groups,

Theorem 4.6. *[Smi83, Theorem 1.6] Let $G \in \mathcal{H}$ with $G = \langle x, y, z | [x, y] = z^t, [x, z] = [y, z] = 1\rangle$. If the p^{th}-part of t is p^{θ} then*

$$\zeta_G^{\triangleleft}(s) = (1 - p^{\theta(2-s)})\zeta_{C_{\infty}^d}^{\triangleleft}(s) + p^{\theta(2-s)}\zeta_{F_2^2}^{\triangleleft}(s).$$

Corollary 4.7. *Let $G \in \mathcal{H}$ then $\alpha_G^{\triangleleft} = 2$.*

4.2 Co-normal zeta functions

We now introduce a new kind of co-index zeta function.

Definition. *Let G be a finitely generated torsion free nilpotent group. The* normal co-index (or co-normal) zeta function *of G is*

$$\zeta_G^{\triangleleft,\mathrm{up}}(s) = \sum_{\substack{G \le T \le G^{\mathbb{Q}} \\ |T:G|<\infty \\ G \trianglelefteq T}} |T:G|^{-s}$$

where $G^{\mathbb{Q}}$ is the Mal'cev completion of G.

We aim to calculate the normal co-index zeta function explicitly for groups which we have co-index zeta functions. Firstly, if $G = \mathbb{Z}^n$ we have $G^{\mathbb{Q}} \simeq \mathbb{Q}^n$ which is abelian and so the following is immediate.

Proposition 4.8. $\zeta_{\mathbb{Z}^n}^{\triangleleft,\mathrm{up}}(s) = \zeta_{\mathbb{Z}^n}^{\mathrm{up}}(s) = \zeta_{\mathbb{Z}^n}(s)$.

Corollary 4.9. *Let $G = \mathbb{Z}^n$ then $\alpha^{\mathrm{up},\triangleleft} = n$.*

(Where as always $\alpha^{\mathrm{up},\triangleleft}$ is the abscissa of convergence of $\zeta^{\mathrm{up},\triangleleft}(s)$.)

We next observe that we have an Euler product for $\zeta_{\mathbb{Z}^n}^{\triangleleft,\mathrm{up}}(s)$. The proof is as that of the Euler product for $\zeta_G^{\mathrm{up}}(s)$ and we omit the details.

Lemma 4.10. *Let $G \in \mathcal{T}$, then we have*

$$\zeta_G^{\triangleleft,\mathrm{up}}(s) = \prod_{p\in\mathcal{P}} \zeta_G^{p,\triangleleft,\mathrm{up}}(s)$$

where

$$\zeta_G^{p,\triangleleft,\mathrm{up}}(s) = \sum_{\substack{G \trianglelefteq_f T \\ |T:G| \text{ a } p\text{-power}}} |T:G|^{-s}.$$

4.3 Co-normal zeta function of $\mathcal{H}$-groups

We now consider $\zeta_G^{\triangleleft,\mathrm{up}}(s)$ for $\mathcal{H}$-groups. We have already determined conditions which allow us to calculate $\zeta_G^{\mathrm{up}}(s)$ for such groups. Hence we must determine when a finite co-index overgroup T of $G \in \mathcal{H}$ has the property that $G \trianglelefteq T$.

Lemma 4.11. *Let $G \in \mathcal{H}$, i.e. $G = \langle x,y,z | [x,y] = z^t, [x,z] = [y,z] = 1\rangle$ for* *some $t \in \mathbb{N}$. Let $G \leq T \leq G^{\mathbb{Q}}$ with T described by the good basis,*

$$\begin{array}{ccc} x^{\alpha^{-1}} & y^{\beta} & z^{\gamma} \\ & y^{\delta^{-1}} & z^{\varepsilon} \\ & & z^{\lambda^{-1}} \end{array}$$

(note that we say nothing about the values of $\alpha, \beta, \gamma, \delta, \varepsilon, \lambda$ we only assert that the above is a good basis). If

$$z^{t\alpha^{-1}}, z^{t\beta}, z^{t\delta^{-1}} \in G$$

then $G \trianglelefteq T$.

Proof. If $G \trianglelefteq T$ then for each generator g of G and h of T we must have $g^h \in G$ or equivalently $[g, h] \in G$. This is what we now check.

Firstly note that z is central and so we need not check generators just involving powers of z. Thus we only have to consider when

$$[x, x^{\alpha^{-1}} y^{\beta} z^{\gamma}], [x, y^{\delta^{-1}} z^{\varepsilon}], [y, x^{\alpha^{-1}} y^{\beta} z^{\gamma}] \in G.$$

So,

$$\begin{aligned} [x, x^{\alpha^{-1}} y^{\beta} z^{\gamma}] &= (x^{-1})(x^{-\alpha^{-1}} y^{-\beta} z^{-\gamma - t\alpha^{-1}\beta})(x)(x^{\alpha^{-1}} y^{\beta} z^{\gamma}) \\ &= (x^{-(\alpha^{-1}+1)} y^{-\beta} z^{-\gamma - t\alpha^{-1}\beta})(x^{(\alpha^{-1}+1)} y^{\beta} z^{\gamma}) \\ &= z^{-t\alpha^{-1}\beta} \cdot z^{-t(-\beta(\alpha^{-1}+1))} \\ &= z^{t\beta} \end{aligned}$$

$$\begin{aligned} [x, y^{\delta^{-1}} z^{\varepsilon}] &= (x^{-1})(y^{-\delta^{-1}} z^{-\varepsilon})(x)(y^{\delta^{-1}} z^{\varepsilon}) \\ &= z^{t\delta^{-1}} \end{aligned}$$

$$\begin{aligned} [y, x^{\alpha^{-1}} y^{\beta} z^{\gamma}] &= (y^{-1})(x^{-\alpha^{-1}} y^{-\beta} z^{-\gamma - t\alpha^{-1}\beta})(y)(x^{\alpha^{-1}} y^{\beta} z^{\gamma}) \\ &= (x^{-\alpha^{-1}} y^{-(\beta+1)} z^{-\gamma - t\alpha^{-1}\beta - t\alpha^{-1}})(x^{\alpha^{-1}} y^{1+\beta} z^{\gamma - t\alpha^{-1}}) \\ &= z^{(-t\alpha^{-1}\beta - 2t\alpha^{-1} + t(\beta+1)\alpha^{-1})} \\ &= z^{-t\alpha^{-1}}. \end{aligned}$$

Hence we need $z^{-t\alpha^{-1}}, z^{t\beta}, z^{t\delta^{-1}} \in G$. $\square$

We now use this lemma to calculate the normal co-index zeta functions for these groups.

Theorem 4.12. *Let G be the Heisenberg Group (i.e. $G = F_2^2 = \langle x, y, z | [x, y] = z, [x, z] = [y, z] = 1\rangle$), then*

$$\zeta_G^{\lhd,\mathrm{up}}(s) = \zeta(s).$$

Corollary 4.13. *Let $G = F_2^2$ then $\alpha_G^{up,\lhd} = 1$.*

Proof. We know that if $T \geq G$ then T is given by a good basis

$$\begin{array}{ccc} x^{\alpha^{-1}} & y^{\beta} & z^{\gamma} \\ & y^{\delta^{-1}} & z^{\varepsilon} \\ & & z^{\lambda^{-1}} \end{array}$$

where

$$\alpha, \delta, \lambda \in \mathbb{N} \tag{4.1}$$
$$\lambda = l\alpha\delta \text{ for some } l \in \mathbb{Z} \tag{4.2}$$
$$0 < \beta \leq \delta^{-1} \text{ and } \alpha\beta\delta \in \mathbb{Z} \tag{4.3}$$
$$0 < \varepsilon \leq \lambda^{-1} \text{ and } \varepsilon\delta\lambda \in \mathbb{Z} \tag{4.4}$$
$$0 < \gamma \leq \delta^{-1} \text{ and } \gamma = \tfrac{1}{\alpha\lambda}\left(c + \tbinom{\alpha}{2}\tfrac{\beta\lambda}{\alpha} + \beta\delta\varepsilon\lambda\right) \text{ for some } c \in \mathbb{Z}. \tag{4.5}$$

We now use Lemma 4.11 to determine when such a T has the property that $G \trianglelefteq T$. So we need

$$z^{\alpha^{-1}}, z^{\beta}, z^{\delta^{-1}} \in G$$

(as $t = 1$). This means that we must have $\alpha^{-1}, \beta, \delta^{-1} \in \mathbb{Z}$. As $\alpha \in \mathbb{N}$ and $\alpha^{-1} \in \mathbb{Z}$ we must have $\alpha = 1$. Similarly we must have $\delta = 1$.

From condition (4.3) we see that we must have

$$0 < \beta \leq \delta^{-1} = 1 \text{ and } \alpha\beta\delta = \beta \in \mathbb{Z},$$

in addition to the fact that $\beta \in \mathbb{Z}$ to force $T \trianglerighteq G$. Thus we can only choose $\beta = 1$.

For condition (4.4) where we have

$$0 < \varepsilon \leq \lambda^{-1} \text{ and } \varepsilon\lambda \in \mathbb{Z}.$$

We firstly note that we may set $\varepsilon = \frac{a}{\lambda}$ for some $a \in \mathbb{Z}$ and then as

$$0 < \varepsilon = \frac{a}{\lambda} \leq \lambda^{-1}$$

the only valid choice we have is $a = 1$. Now we need to consider the valid choices for γ. We have,

$$\begin{aligned}
\gamma &= \frac{1}{\alpha\lambda}\left(c + \binom{\alpha}{2}\frac{\beta\lambda}{\alpha} + \beta\delta\varepsilon\lambda\right) \\
&= \frac{1}{\lambda}\left(c + \lambda \cdot \frac{1}{\lambda}\right) \\
&= \frac{1}{\lambda}(c+1)
\end{aligned}$$

and

$$0 < \gamma = \frac{1}{\lambda}(c+1) \leq \lambda^{-1}$$

so we need

$$0 < c + 1 \leq 1.$$

Thus we are forced to fix $c = 0$ and so have a single choice for c.

Hence we must have a good basis as follows,

$$\begin{array}{ccc}
x & y & z^{\gamma} \\
 & y & z^{\frac{1}{\lambda}} \\
 & & z^{\lambda^{-1}}
\end{array}$$

where γ is fixed (as above) and $\lambda \in \mathbb{Z}$. Hence we have a single co-normal overgroup of each co-index so that

$$\zeta_H^{\triangleleft,\mathrm{up}}(s) = \zeta(s).$$

□

Theorem 4.14. *Let $G \in \mathcal{H}$ with $G = \langle x, y, z | [x,y] = z^t, [x,z] = [y,z] = 1\rangle$ (with $t \in \mathbb{N}$ and $t > 1$), then*

$$\zeta_G^{\triangleleft,\mathrm{up}}(s) = t^{(2-3s)}\zeta(s)$$

Corollary 4.15. *Let $G \in \mathcal{H}$ then $\alpha_G^{up,\lhd} = 1$.*

Proof. As above we have that if $T \geq G$ then T is determined by a good basis,

$$\begin{matrix} x^{\alpha^{-1}} & y^{\beta} & z^{\gamma} \\ & y^{\delta^{-1}} & z^{\varepsilon} \\ & & z^{\lambda^{-1}} \end{matrix}$$

such that

$$\alpha, \delta, \lambda \in \mathbb{N} \tag{4.6}$$
$$t\lambda = l\alpha\delta \text{ for some } l \in \mathbb{Z} \tag{4.7}$$
$$0 < \beta \leq \delta^{-1} \text{ and } \alpha\beta\delta \in \mathbb{Z} \tag{4.8}$$
$$0 < \varepsilon \leq \lambda^{-1} \text{ and } \varepsilon\delta\lambda \in \mathbb{Z} \tag{4.9}$$
$$0 < \gamma \leq \delta^{-1} \text{ and } \gamma = \tfrac{1}{\alpha\lambda}\left(c + t\tbinom{\alpha}{2}\tfrac{\beta\lambda}{\alpha} + \beta\delta\varepsilon\lambda\right) \text{ for some } c \in \mathbb{Z} \tag{4.10}$$

We now impose the conditions of Lemma 4.11, that is we insist that

$$z^{t\alpha^{-1}}, z^{t\beta}, z^{t\delta^{-1}} \in G$$

so that $G \unlhd T$.

As $\alpha \in \mathbb{N}$ and as the above conditions mean that $t\alpha^{-1} \in \mathbb{Z}$ we must have $\alpha = t$. Similarly we have that $\delta = t$. If we now consider condition (4.7) this becomes

$$t\lambda = lt^2 \iff \lambda = lt$$

for some $l \in \mathbb{Z}$. We must also ensure that $z^{t\beta} \in G$, in other words, $t\beta \in \mathbb{Z}$. We also note condition (4.8) that is

$$0 < \beta \leq \delta^{-1} \text{ and } \alpha\beta\delta = t^2\beta \in \mathbb{Z}.$$

Hence, as if $t\beta \in \mathbb{Z}$ then $t^2\beta \in \mathbb{Z}$, we can just consider when $t\beta \in \mathbb{Z}$, in other words when $\beta = \frac{a}{t}$ for some $a \in \mathbb{Z}$. Thus,

$$0 < \frac{a}{t} \leq \delta^{-1} = \frac{1}{t} \iff 0 < a \leq 1$$

which forces $a = 1$ and thus $\beta = \frac{1}{t}$.

We next check condition (4.9), which in light of the above becomes,

$$0 < \varepsilon \leq \lambda^{-1} \text{ and } t\varepsilon\lambda \in \mathbb{Z}.$$

As $t\varepsilon\lambda \in \mathbb{Z}$ we will have $\varepsilon = \frac{b}{t\lambda}$ for some $b \in \mathbb{Z}$ so that

$$0 < \frac{b}{t\lambda} \leq \lambda^{-1} \iff 0 < b \leq t.$$

Finally we check condition (4.10),

$$\begin{aligned}
\gamma &= \frac{1}{\alpha\lambda}\left(c + t\tbinom{\alpha}{2}\frac{\beta\lambda}{\alpha} + \beta\delta\varepsilon\lambda\right) \\
&= \frac{1}{t\lambda}\left(c + t\tbinom{t}{2}\frac{\lambda}{t^2} + \frac{\lambda b t}{\lambda t^2}\right) \\
&= \frac{1}{t\lambda}\left(c + \frac{t(t)(t-1)\lambda}{2t^2} + \frac{b}{t}\right) \\
&= \frac{1}{t\lambda}\left(c + \frac{(t-1)\lambda}{2} + \frac{b}{t}\right).
\end{aligned}$$

So we need that

$$0 < \gamma \leq \frac{1}{\lambda}, \quad \text{i.e.} \quad 0 < c + \frac{(t-1)\lambda}{2} + \frac{b}{t} \leq t.$$

We wish to count the valid choices of c and so we wish to know $|(\mathbb{Z}+d) \cap [0,t)|$, where $d = \frac{(t-1)\lambda}{2} + \frac{b}{t}$. This number is t and so we have t choices for γ. So for an overgroup $T \geq G$ to have the property $T \trianglerighteq G$ we must have a good basis,

$$\begin{matrix}
x^{t^{-1}} & y^{t^{-1}} & z^{\gamma} \\
 & y^{t^{-1}} & z^{\varepsilon} \\
 & & z^{\lambda^{-1}}
\end{matrix}$$

where $\lambda = lt$ for some $l \in \mathbb{Z}$, and we have t valid choices for each of ε and γ. Hence

$$\begin{aligned}
\zeta_G^{\triangleleft,\mathrm{up}}(s) &= \sum_{l \in \mathbb{Z}} (lt \cdot t \cdot t)^{-s} t^2 \\
&= \sum_{l \in \mathbb{Z}} t^{(2-3s)} l^{-s} \\
&= t^{(2-3s)} \zeta(s).
\end{aligned}$$

□

4.4 Co-normal Zeta Function of F_n^2

We now attempt to generalise these results to obtain $\zeta_{F_n^c}^{\lhd,\mathrm{up}}(s)$ for certain values of c and n. (Recall that F_n^c is the free nilpotent group on n generators of class c, and that the upper and lower central series of these groups coincide). To begin with, let $G = F_n^c$, so that G has central series,

$$1 \trianglelefteq G_1 \trianglelefteq G_2 \trianglelefteq \cdots \trianglelefteq G_c = G.$$

We note that $G_1 \simeq \mathbb{Z}^d$ for some $d \in \mathbb{N}$. Let $G^{\mathbb{Q}}$ be the Mal'cev completion of G. We then have, using the obvious notation, upper central series

$$1 \trianglelefteq G_1^{\mathbb{Q}} \trianglelefteq G_2^{\mathbb{Q}} \trianglelefteq \cdots \trianglelefteq G_c^{\mathbb{Q}} = G^{\mathbb{Q}}.$$

for $G^{\mathbb{Q}}$ with $G_1^{\mathbb{Q}}$ being the centre of $G^{\mathbb{Q}}$.

Lemma 4.16. *Let G be as above, and let $G_1 \leq_f Z \leq G_1^{\mathbb{Q}}$. Then $T = \langle G, Z \rangle = GZ$ is such that $G \trianglelefteq_f T$.*

(Since $|GZ : G| = |Z : Z \cap G|$, and we have that $Z \cap G \geq G_1$, it must be the case that $|Z : Z \cap G| < \infty$. Clearly $G \trianglelefteq T$.)

We recall the following definition from [Smi83],

Definition. *If*

$$f(s) = \sum_{n=1}^{\infty} a_n n^{-s} \text{ and } g(s) = \sum_{n=1}^{\infty} b_n n^{-s}$$

with $a_i, b_i \in \mathbb{R}$, and $a_i, b_i \geq 0$ for all $i \in \mathbb{N}$ then if $a_i \leq b_i$ we write $f(s) \lessdot g(s)$.

Thus we have, from the Lemma 4.16 above:

Corollary 4.17.

$$\zeta_{\mathbb{Z}^d}^{\lhd,\mathrm{up}}(s) \lessdot \zeta_{F_n^c}^{\lhd,\mathrm{up}}(s)$$

where $d = \mathrm{rank}\ Z(F_n^c)$.

(in fact this is true of any $\mathcal{T}$-group). We note in passing that for the Heisenberg group (which is F_2^2) we have already shown that

$$\zeta_{F_2^2}^{\lhd,\mathrm{up}}(s) = \zeta_{\mathbb{Z}}(s)$$

and extend this result.

Theorem 4.18. *Let F_n^2 be the free nilpotent group of class 2 on n generators. Then*

$$\zeta^{\lhd,\mathrm{up}}_{F_n^2}(s) = \zeta^{\lhd,\mathrm{up}}_{\mathbb{Z}^d}(s)$$

where $d = \mathrm{rank}\ Z(F_n^2)$.

Corollary 4.19. *If $G = F_2^c$ and $d = \mathrm{rank}(Z(F_n^2)) = d$ then $\alpha_G^{up,\lhd} = d$.*

Proof. Let $G = F_n^2 = \langle x_1, x_2, \ldots, x_n | [x_i, x_j] = z_{i,j}, z_{i,j} \text{ central}, 1 \le i < j \le n\rangle$ (note that there are $d = (n-1)+(n-2)+\cdots+1 = \frac{(n-1)(n-2)}{2}$ of the $z_{i,j}$). If T is an overgroup of G of finite co-index, we have a good basis $t_{n+d}, t_{n+d-1}, \ldots, t_d, \ldots, t_1$ such that,

$$\begin{array}{lllllllll}
t_{n+d} = & x_n^{t_{n+d,n+d}} & x_{n-1}^{t_{n+d,n+d-1}} & \cdots & x_1^{t_{n+d,d+1}} & z_{1,2}^{t_{n+d,d}} & \cdots & z_{n-1,n}^{t_{n+d,1}} \\
t_{n+d-1} = & & x_{n-1}^{t_{n+d-1,n+d-1}} & \cdots & x_1^{t_{n+d-1,d+1}} & z_{1,2}^{t_{n+d-1,d}} & \cdots & z_{n-1,n}^{t_{n+d-1,1}} \\
\vdots & & & \ddots & & \vdots & & \vdots \\
t_d = & & & & & z_{1,2}^{t_{d,d}} & \cdots & z_{n-1,n}^{t_{d,1}} \\
\vdots & & & & & & & \vdots \\
t_1 = & & & & & & & z_{n-1,n}^{t_{1,1}}
\end{array}$$

which meet the following conditions:

1. The sequence $t_{n+d}, t_{n+d-1}, \ldots, t_1$ is reduced
2. $T_j = \{t_j^{a_j} t_{j-1}^{a_{j-1}} \cdots t_1^{a_1} | a_i \in \mathbb{Z}\}$ is a group for $j = 1, \ldots, n+d$
3. If $t_j = x_j^{t_{j,j}} x_{j-1}^{t_{j,j-1}} \cdots z_1^{t_{j,1}}$ then $t_{j,j} = \frac{1}{\alpha_j}$ for some $\alpha_j \in \mathbb{N}$
4. $x_j^{-1} t_j^{\alpha_j} \in T_{j-1}$.

We note that $0 < t_{k,j} \le \frac{1}{\alpha_j}$ for $n \ge k > j$ as the $t_{n+d}, t_{n+d-1}, \ldots, t_1$ is reduced.

If $G \trianglelefteq T$, then $[G, T] \le G$. As both T and G are finitely generated, we test that $[g, t] \in G$ for each generator g of G and t of T. We note that

$$[z_j, t_k] = [x_i, t_l] = 1$$

as z_j and t_l are central for $1 \le i < j \le n$, $1 \le k \le n$ and $d \ge l \ge 1$.

Recalling P. Hall's commutator formulas,

$$[x, yz] = [x, z][x, y]^z$$
$$[xy, z] = [x, z]^y[y, z]$$

we first check that $[x_2, t_{d+1}] \in G$. For brevity we write,

$$t_{d+1} = x_1^{t_{d+1,d+1}} z_{1,2}^{t_{d+1,d}} z_{1,3}^{t_{d+1,d-1}} \cdots z_{n-1,n}^{t_{d+1,1}}$$

as

$$t_{d+1} = x_1^{\alpha^{-1}} \underline{z}^{\underline{\gamma}}$$

where $t_{d+1,d+1} = \frac{1}{\alpha}$ and $\underline{z}^{\underline{\gamma}} = z_{1,2}^{t_{d+1,d}} z_{1,3}^{t_{d+1,d-1}} \cdots z_{n-1,n}^{t_{d+1,1}}$. Hence,

$$\begin{aligned}
[x_2, t_{d+1}] &= [x_2, x_1^{\alpha^{-1}} \underline{z}^{\underline{\gamma}}] \\
&= [x_2, \underline{z}^{\underline{\gamma}}]^{x_1^{\alpha^{-1}}} [x_2, x_1^{\alpha^{-1}}] \\
&= [x_2, x_1]^{\alpha^{-1}} = z_{1,2}^{-\alpha^{-1}}
\end{aligned}$$

since $\underline{z}^{\underline{\gamma}}$ is central. Hence we need $\alpha^{-1} \in \mathbb{Z}$ if $[x_2, t_{d+1}] \in G$. We also require that $\alpha \in \mathbb{N}$ so we see that $\alpha = 1$. We now check that

$$x_1^{-1} \cdot t_{d+1}^1 \in T_d = \langle t_d, t_{d-1} \ldots, t_1 \rangle.$$

That is we require

$$\underline{z}^{\underline{\gamma}} = z_{1,2}^{t_{d+1,d}} z_{1,3}^{t_{d+1,d-1}} \cdots z_{n-1,n}^{t_{d+1,1}} = t_d^{\alpha_d} t_{d-1}^{\alpha_{d-1}} \cdots t_1^{\alpha_1}$$

for some $\alpha_j \in \mathbb{Z}$. From the above conditions, we see that we must have

$$\begin{aligned}
0 < t_{d+1,d} &\leq t_{d,d} \\
\text{and} \quad t_{d+1,d} &= \alpha_d \cdot t_{d,d}
\end{aligned}$$

forcing us to choose $\alpha_d = 1$ and $t_{d+1,d}$ to be fixed at $t_{d,d}$. Next we must have,

$$\begin{aligned}
0 < t_{d+1,d-1} &\leq t_{d-1,d-1} \\
\text{and} \quad t_{d+1,d_1} &= t_{d,d-1} + \alpha_{d-1} \cdot t_{d-1,d-1}
\end{aligned}$$

observing that as $0 < t_{d+1,d-1} \leq t_{d-1,d-1}$ we will have,

$$-t_{d-1,d-1} < t_{d+1,d_1} - t_{d,d-1} < t_{d-1,d-1}$$

and so we are forced to select $\alpha_{d-1} = 0$ so that $t_{d+1,d-1}$ is fixed. Similar arguments allow us to see that $t_{d+1,j}$ is then fixed for $1 \leq j \leq d$.

Next we consider the extra conditions necessary for $[x_3, t_{d+2}] \in G$. Again we write

$$t_{d+2} = x_2^{\alpha^{-1}} x_1^{\beta} \underline{z}^{\underline{\gamma}}$$

and apply the above commutator formulas to see that

$$\begin{aligned}
[x_3, t_{d+2}] &= [x_3, x_2^{\alpha^{-1}} x_1^{\beta} \underline{z}^{\underline{\gamma}}] \\
&= [x_3, x_2^{\alpha^{-1}} x_1^{\beta}] \cdot [x_3, \underline{z}^{\underline{\gamma}}]^{x_2^{\alpha^{-1}} x_1^{\beta}} \\
&= [x_3, x_2^{\alpha^{-1}} x_1^{\beta}] \text{ as } [\cdot,\cdot] \text{ is central} \\
&= [x_3, x_1^{\beta}][x_3, x_2^{\alpha^{-1}}] \\
&= z_{1,3}^{-\beta} z_{2,3}^{-\alpha^{-1}}.
\end{aligned}$$

This forces $\alpha = 1$ (as both $\alpha^{-1} \in \mathbb{Z}$ and $\alpha \in \mathbb{N}$ must hold) and as $\beta \in \mathbb{Z}$ and $0 < \alpha_1 \leq 1$ we also have $\beta = 1$. We now consider when,

$$x_2^{-1} t_{d+2} \in T_{d+1}$$

that is when is $x_1 \underline{z}^{\underline{\gamma}} \in T_{d+1}$. As above we argue by considering the exponents one at a time and see that each is fixed.

We follow this procedure and continue to refine our good basis to see that we have a single choice of t_i for $n \geq i > d$, that each has leading coefficient 1. However we are free to choose $t_d, \cdots t_1$ and hence,

$$\zeta_{F_n^2}^{\mathrm{up},\triangleleft}(s) = \zeta_{\mathbb{Z}^d}(s).$$

$\square$

We will now calculate $\zeta_{F_3^2}^{\mathrm{up},\triangleleft}(s)$ to give an example of this proof. Let $F_3^2 = \langle w, x, y | [w,x] = z_1, [w,y] = z_2, [x,y] = z_3, z_i \text{ central} \rangle$. An overgroup $T \geq F_3^2$ will have a good basis

$$\begin{array}{cccccccl}
w^{a^{-1}} & x^b & y^c & z_1^d & z_2^e & z_3^f & = & t_6 \\
 & x^{g^{-1}} & y^h & z_1^i & z_2^j & z_3^k & = & t_5 \\
 & & y^{l^{-1}} & z_1^m & z_2^n & z_3^o & = & t_4 \\
 & & & z_1^{p^{-1}} & z_2^q & z_3^r & = & t_3 \\
 & & & & z_2^{s^{-1}} & z_3^t & = & t_2 \\
 & & & & & z_3^{u^{-1}} & = & t_1
\end{array}$$

which must satisfy

1. $t_6, t_5, \ldots, t_1$ reduced
2. T_i, for $i = 1, 2, \ldots 6$, is a group
3. $a, g, l, p, s, u \in \mathbb{N}$
4. $w^{-1}t_6^a \in T_5, \ldots, z_2^{-1}t_2^s \in T_1$.

We assume that we have already met these conditions. We now check to see if $F_3^2 \trianglelefteq T$, by considering the appropriate commutators. Firstly,

$$\begin{aligned}
[x, t_4] &= [x, y^{l^{-1}} z_1^m z_2^n z_3^o] \\
&= [x, z_1^m z_2^n z_3^o][x, y^{l^{-1}}]^{z_1^m z_2^n z_3^o}
\end{aligned}$$

which as the z_i are central,

$$\begin{aligned}
&= [x, y^{l^{-1}}] \\
&= z_3^{l^{-1}}.
\end{aligned}$$

Thus we must have $l^{-1} \in \mathbb{Z}$ if $z_3^{l^{-1}} \in F_3^2$. As $l \in \mathbb{N}$ we have $l = 1$. It remains to check that

$$y^{-1}(yz_1^m z_2^n z_3^o) \in T_3.$$

That is we must have,

$$z_1^i z_2^j z_3^k = t_3^\alpha t_2^\beta t_1^\gamma$$

so we need

$$\begin{aligned}
m &= \alpha p^{-1} \\
n &= \alpha q + \beta s^{-1} \\
o &= \alpha r + \beta t + \gamma u^{-1}.
\end{aligned}$$

As the sequence $t_i, t_{i-1}, \ldots t_1$ is reduced we know that in particular for t_i we have $0 < m \leq p^{-1}$ and as $\alpha p^{-1} = m$ we must have $\alpha = 1$ and so $m = \frac{1}{p}$ is fixed. We now need,

$$\begin{aligned}
n &= q + \beta s^{-1} \\
o &= r + \beta t + \gamma u^{-1}.
\end{aligned}$$

Again, as the sequence $t_i, t_{i-1}, \ldots t_1$ is reduced we know that, in particular, for t_i we have

$$0 < n \leq s^{-1}$$
$$0 < q \leq s^{-1}$$

and so we have

$$0 < q + \beta s^{-1} \leq s^{-1}$$

consequently,

$$0 < qs + \beta \leq 1.$$

We know that $0 < qs \leq 1$ and as $\beta \in \mathbb{Z}$ we are forced into choosing $\beta = 0$. Hence we have a fixed choice of $n = q$. Finally we have

$$o = r + \gamma u^{-1}$$

with

$$0 < o \leq u^{-1}$$
$$0 < r \leq u^{-1}$$

so that $0 < r + \gamma u^{-1} \leq u^{-1}$ or $0 < ru + \gamma \leq 1$, again forcing us to select $\gamma = 0$

and $o = r$. Thus we have that t_4 is now fixed and our good basis now looks like,

$$\begin{array}{cccccccc}
w^{a^{-1}} & x^b & y^c & z_1^d & z_2^e & z_3^f & = & t_6 \\
 & x^{g^{-1}} & y^h & z_1^i & z_2^j & z_3^k & = & t_5 \\
 & & y & z_1^{p^{-1}} & z_2^q & z_3^r & = & t_4 \\
 & & & z_1^{p^{-1}} & z_2^q & z_3^r & = & t_3 \\
 & & & & z_2^{s^{-1}} & z_3^t & = & t_2 \\
 & & & & & z_3^{u^{-1}} & = & t_1.
\end{array}$$

We now consider,

$$\begin{aligned}
[w, t_5] &= [w, x^{g^{-1}} y^h z_1^i z_2^j z_3^k] \\
&= [w, z_1^i z_2^j z_3^k][w, x^{g^{-1}} y^h]^{z_1^i z_2^j z_3^k}
\end{aligned}$$

which as $z_1^i z_2^j z_3^k$ is central we have,

$$\begin{aligned}
&= [w, x^{g^{-1}} y^h] \\
&= [w, y^h][w, x^{g^{-1}}]^{y^h} \\
&= z_2^h z_1^{g^{-1}}.
\end{aligned}$$

If $z_2^h z_1^{g^{-1}}$ this is to belong to F_3^2 we must have $g = 1$ (as $g^{-1} \in \mathbb{Z}$ and $g \in \mathbb{N}$) and $h = 1$ (as $h \in \mathbb{Z}$ and $0 < h \leq 1$). Hence t_5 now looks like

$$xyz_1^i z_2^j z_3^k.$$

We now must have $x^{-1}t_5 \in T_4$ so that

$$yz_1^i z_2^j z_3^k = t_4^\alpha t_3^\beta t_2^\gamma t_1^\delta.$$

So we need,

$$\begin{aligned}
\alpha &= 1 \\
i &= \alpha p^{-1} + \beta p^{-1} \\
j &= \alpha q + \beta q + \gamma s^{-1} \\
k &= \alpha r + \beta r + \gamma t + \delta u^{-1}
\end{aligned}$$

which simplifies to

$$\begin{aligned}
i &= (\beta+1)p^{-1} \\
j &= (\beta+1)q+\gamma s^{-1} \\
k &= (\beta+1)r+\gamma t+\delta u^{-1}.
\end{aligned}$$

Now, $0 < i \leq p^{-1}$ and hence $(\beta+1) = 1$ so that $\beta = 0$ and i is fixed. Thus we have,

$$j = q + \gamma s^{-1}$$

with $0 < j \leq s^{-1}$ and $0 < q \leq s^{-1}$ so that we have

$$0 < qs + \gamma \leq 1.$$

Therefore $\gamma = 0$ so that $j = q$ is fixed. Finally, and with a similar calculation have

$$k = r + \delta u^{-1}$$

so that $0 < ru + \delta \leq 1$ which forces $\delta = 0$ and so $k = r$. With this our good basis now looks like,

$$\begin{array}{ccccccccc}
w^{a^{-1}} & x^b & y^c & z_1^d & z_2^e & z_3^f & = & t_6 \\
 & x & y & z_1^{p^{-1}} & z_2^q & z_3^r & = & t_5 \\
 & & y & z_1^{p^{-1}} & z_2^q & z_3^r & = & t_4 \\
 & & & z_1^{p^{-1}} & z_2^q & z_3^r & = & t_3 \\
 & & & & z_2^{s^{-1}} & z_3^t & = & t_2 \\
 & & & & & z_3^{u^{-1}} & = & t_1.
\end{array}$$

Finally we consider

$$\begin{aligned}
[x, t_6] &= [x, w^{a^{-1}} x^b y^c z_1^d z_2^e z_3^f] \\
&= [x, z_1^d z_2^e z_3^f][x, w^{a^{-1}} x^b y^c]^{z_1^d z_2^e z_3^f} \\
&= [x, w^{a^{-1}} x^b y^c],
\end{aligned}$$

and as $z_1^d z_2^e z_3^f$ is central,

$$\begin{aligned}
&= [x, x^b y^c][x, w^{a^{-1}}]^{x^b y^c} \\
&= [x, y^c][x, x^b]^{y^c}[x, w^{a^{-1}}]^{x^b y^c} \\
&= z_1^{a^{-1}} z_3^c.
\end{aligned}$$

As $z_1^{a^{-1}} z_3^c \in F_3^2$ if $F_3^2 \trianglelefteq T$ we must have $a = 1$ (as $a \in \mathbb{N}$ and $a^{-1} \in \mathbb{Z}$) and $c = 1$ (as $c \in \mathbb{Z}$ and $0 < c \leq 1$). Now we must have $w^{-1}t_6 \in T_5$ so that,

$$x^b y z_1^d z_2^f z_3^e = t_5^\alpha t_4^\beta t_3^\gamma t_2^\delta t_1^\varepsilon$$

so that

$$\begin{aligned} b &= \alpha \\ 1 &= \alpha + \beta \\ d &= p^{-1}(\alpha + \beta + \gamma) \\ e &= q(\alpha + \beta + \gamma) + \delta s^{-1} \\ f &= r(\alpha + \beta + \gamma) + \delta t + \varepsilon u^{-1}. \end{aligned}$$

As $\alpha \in \mathbb{Z}$ and $0 < b \leq 1$ we must have $b = 1$ so that $\beta = 0$. Hence the above simplifies to

$$\begin{aligned} d &= (\gamma + 1)p^{-1} \\ e &= q(\gamma + 1) + \delta s^{-1} \\ f &= r(\gamma + 1) + \delta t + \varepsilon u^{-1}. \end{aligned}$$

As $0 < d \leq p^{-1}$ we must have $\gamma = 0$. Now as $0 < e \leq s^{-1}$ we have $0 < qs + \delta \leq 1$, forcing $\delta = 0$ as $0 < qs \leq 1$. Similarly, $\varepsilon = 0$ leaving us with a good basis,

$$\begin{array}{cccccccc} w & x & y & z_1^{p^{-1}} & z_2^q & z_3^r & = & t_6 \\ & x & y & z_1^{p^{-1}} & z_2^q & z_3^r & = & t_5 \\ & & y & z_1^{p^{-1}} & z_2^q & z_3^r & = & t_4 \\ & & & z_1^{p^{-1}} & z_2^q & z_3^r & = & t_3 \\ & & & & z_2^{s^{-1}} & z_3^t & = & t_2 \\ & & & & & z_3^{u^{-1}} & = & t_1 \end{array}$$

where we are free to choose p, q, r, s, t, u provided they do not break the above constraints. Thus we have,

$$\zeta_{F_3^2}^{\mathrm{up},\triangleleft}(s) = \zeta(s)\zeta(s-1)\zeta(s-2).$$

4.5 Good Bases Revisited

We make the following observation concerning the good basis for the overgroup $T \trianglerighteq F_3^2$, it is a good basis but it is not the most attractive in this context. If we replace t_4 by $t_4' = t_4 t_3^{-1}$, t_5 by $t_5' = t_5(t_4')^{-1}t_3^{-1}$, and finally t_6 by $t_6' = t_6(t_5')^{-1}(t_4')^{-1}t_3^{-1}$, we obtain the following good basis

$$\begin{array}{cccccccl}
w & & & & & & = & t_6 \\
 & x & & & & & = & t_5 \\
 & & y & & & & = & t_4 \\
 & & & z_1^{p-1} & z_2^{q} & z_3^{r} & = & t_3 \\
 & & & & z_2^{s-1} & z_3^{t} & = & t_2 \\
 & & & & & z_3^{u-1} & = & t_1
\end{array}$$

which facilitates calculation and we can also see that $T = \langle F_2^3, Z\rangle$ for some Z with $Z(F_2^3) \leq Z \leq Z(F_2^3)^{\mathbb{Q}}$.

The reason for the effectiveness of the above manoeuvre is the order that we use to determine when a sequence of elements is reduced. Recall that we are using the order $\sqsubset_{\mathbb{Q}}$, which is defined as follows:

1. Firstly order $\mathbb{Q}$ in the usual manner (which we denote by $<$). For $a, b \in \mathbb{Q}$ we write $a \prec_{\mathbb{Q}} b$ if
   (a) $0 < a, b$ and $a < b$, or
   (b) $0 < a$ and $b < 0$, or
   (c) $a, b \leq 0$ and $b < a$.
2. Order $\mathbb{Q}^n$ lexicographically by $(\lambda_n, \lambda_{n-1} \ldots, \lambda_1) \sqsubset_{\mathbb{Q}} (\mu_n, \mu_{n-1}, \ldots, \mu_1)$ *i.e.* if there is an i such that $\lambda_i \prec_{\mathbb{Q}} \mu_i$ but $\lambda_j = \mu_j$ for $j > i$.

In this ordering zero is not preferred; a situation which leads to bases such as the one above.

What we seek to do is to define a new order $\sqsubset_{\mathbb{Q}}'$ which, after the first non-zero element, preferred zeros. Thus, we define

Definition. *Order $\mathbb{Q}$ is the usual manner (denoted $<$). For $a, b \in \mathbb{Q}$ we write $a <_{\mathbb{Q}} b$ if* *(i)* $0 \leq a, b$ *and* $a < b$, *or*

(ii) $0 \leq a$ *and* $b < 0$, *or*

(iii) $a, b < 0$ *and* $b < a$.

Definition. *We order* $\mathbb{Q}^n$ *lexicographically using* $<_{\mathbb{Q}}$ *by* $(\lambda_n, \lambda_{n-1}, \ldots, \lambda_1) \sqsubset'_{\mathbb{Q}} (\mu_n, \mu_{n-1}, \ldots, \mu_1)$ *if there is an* i *such that* $\lambda_i \prec_{\mathbb{Q}} \mu_i$ *with* $\lambda_j = \mu_j = 0$ *for* $j > i$ *or if there is an* i *such that* $\lambda_i <_{\mathbb{Q}} \mu_i$ *and* $\lambda_j = \mu_j$ *for* $j > i$ *with at least one* μ_j *or* λ_j *non-zero.*

It should be noted that rather than a single ordering as with $\sqsubset_{\mathbb{Q}}$ this is in fact a family of orderings, with a different order on $\mathbb{Q}^i$ for each i. Thus care needs to be taken when using this ordering as it is 'not quite compatible' with multiplication and this is why we have only introduced it at this late point.

Now using the new $\sqsubset'_{\mathbb{Q}}$ order the calculation of the co-normal zeta function is simplified; if we again consider $T \geq F_3^2 = \langle w, x, y | [w, x] = z_1, [w, y] = z_2, [x, y] = z_3, z_i \text{ central}\rangle$, with "good" basis (using our new ordering)

$$\begin{array}{cccccccc}
w^{a^{-1}} & x^b & y^c & z_1^d & z_2^e & z_3^f & = & t_6 \\
 & x^{g^{-1}} & y^h & z_1^i & z_2^j & z_3^k & = & t_5 \\
 & & y^{l^{-1}} & z_1^m & z_2^n & z_3^o & = & t_4 \\
 & & & z_1^{p^{-1}} & z_2^q & z_3^r & = & t_3 \\
 & & & & z_2^{s^{-1}} & z_3^t & = & t_2 \\
 & & & & & z_3^{u^{-1}} & = & t_1
\end{array}$$

and begin to determine if $[F_3^2, T] \leq F_3^2$. Firstly, we consider

$$\begin{aligned}
[x, t_4] &= [x, y^{l^{-1}} \underline{z}^{\underline{\gamma}}] \\
&= [x, \underline{z}^{\underline{\gamma}}][x, y^{l^{-1}}]^{\underline{z}^{\underline{\gamma}}} \\
&= z_3^{l^{-1}}
\end{aligned}$$

and we find that $l = 1$ (as $l \in \mathbb{N}$ and $l^{-1} \in \mathbb{Z}$). We now check that

$$y^{-1} t_4 \in T_3$$

which holds when

$$z_1^m z_2^n z_3^o = t_3^\alpha t_2^\beta t_1^\gamma.$$

Thus we must have

$$\begin{aligned} m &= \alpha p^{-1} \\ n &= \alpha q + \beta s^{-1} \\ o &= \alpha r + \beta t + \gamma u^{-1}. \end{aligned}$$

Now, (and using the new order) we need $0 \leq m < p^{-1}$ and $m = \alpha p^{-1}$ so that $\alpha = m = 0$. This means that as $0 \leq n = \beta s^{\imath} < s^{-1}$ we have $\beta = n = 0$. Finally we must have $o = 0$. Continuing in a similar fashion we end up with T having a good basis

$$\begin{array}{ccccccccc} w & & & & & & & = & t_6 \\ & x & & & & & & = & t_5 \\ & & y & & & & & = & t_4 \\ & & & z_1^{p^{-1}} & z_2^{q} & z_3^{r} & & = & t_3 \\ & & & & z_2^{s^{-1}} & z_3^{t} & & = & t_2 \\ & & & & & z_3^{u^{-1}} & & = & t_1. \end{array}$$

Thus the new ordering renders the calculation for the co-normal function for F_n^2 more straightforward, as we begin to find more zero coefficients on non-leading terms.

4.6 The co-index zeta function of F_c^n

We use the strategy of §4.5 to calculate $\zeta_{F_n^c}^{\triangleleft,\mathrm{up}}(s)$.

Theorem 4.20. *Let F_n^c be the free nilpotent group of class c on n generators. Then*

$$\zeta_{F_n^c}^{\triangleleft,up}(s) = \zeta_{\mathbb{Z}^d}(s)$$

where $d = rank\ Z(F_n^c)$.

Corollary 4.21. *If $G = F_n^c$ and $d = \mathrm{rank}(Z(F_n^c))$ then $\alpha_G^{up,\triangleleft} = d$.*

Proof. Let $F_n^c = \langle x_1, x_2, \ldots, x_n\rangle/\gamma_c(F)$. If T is an overgroup of finite co-index, we have a good basis for T part of which we show below,

$$\begin{array}{lllllllll}
t_1 & & & & & & & & z_1^{t_{1,1}} \\
t_2 & & & & & & & z_2^{t_{2,2}} & z_1^{t_{2,1}} \\
\vdots & & & & & & & & \\
t_d & & & & z_d^{t_{d,d}} & z_{d-1}^{t_{d,d-1}} & \cdots & z_2^{t_{d,2}} & z_1^{t_{d,1}} \\
t_{d+1} & & w_1^{t_{d+1,d+1}} & \cdots & z_d^{t_{d+1,d}} & z_{d-1}^{t_{d+1,d-1}} & \cdots & z_2^{t_{d+1,2}} & z_1^{t_{d+1,1}} \\
t_{d+2} & w_2^{t_{d+2,d+2}} & w_1^{t_{d+2,d+1}} & \cdots & z_d^{t_{d+2,d}} & z_{d-1}^{t_{d+2,d-1}} & \cdots & z_2^{t_{d+2,2}} & z_1^{t_{d+2,1}}
\end{array}$$

where $w_i \in \gamma_{c-1}$ and as always assume that basis meets the condition for it to be a good basis (in the $\sqsubset'_{\mathbb{Q}}$ ordering). As we are dealing with F_n^c we firstly consider when

$$[x_i, t_{d+1}] \in F_n^c \quad \text{where } [x_i, w_1] = z_j$$

For convenience we write $t_{d+1} = w_1^{\alpha^{-1}} \underline{z}^{\underline{\gamma}}$ where $t_{d+1,d+1} = \frac{1}{\alpha}$ and $\alpha \in \mathbb{N}$. So that

$$\begin{aligned}
[x_i, t_{d+1}] &= [x_i, w_1^{\alpha^{-1}} \underline{z}^{\underline{\gamma}}] \\
&= [x_i, \underline{z}^{\underline{\gamma}}] \cdot [x_i, w_1^{\alpha^{-1}}]^{\underline{z}^{\underline{\gamma}}} \\
&= z_j^{\alpha^{-1}}
\end{aligned}$$

as $\underline{z}^{\underline{\gamma}}$ is central. Thus we require $z_j^{\alpha^{-1}} \in G = F_n^c$, which means that $\alpha^{-1} \in \mathbb{Z}$, and as we already have $\alpha \in \mathbb{N}$ we must have $\alpha = 1$. We now check to see if

$$w_1^{-1} t_{d+1} \in T_d$$

that is

$$z_d^{t_{d+1,d}} z_{d-1}^{t_{d+1,d-1}} \cdots z_2^{t_{d+1,2}} z_1^{t_{d+1,1}} = t_d^{\alpha_d} t_{d-1}^{\alpha_{d-1}} \cdots t_2^{\alpha_2} t_1^{\alpha_1}.$$

We know that the sequence $t_j, t_{j-1}, \ldots, t_1$ is reduced (using the above ordering), so we must have $0 \leq t_{d+1,d} < t_{d+1,d+1}$ and also we must have that $\alpha_d \cdot t_{d,d} = t_{d+1,d}$ for the above equation to hold. This forces $t_{d+1,d} = \alpha_d = 0$. Next we have that

$$\begin{aligned}
0 \leq t_{d+1,d_2} &< t_{-1,d-1} \\
\text{and} \quad \alpha_{d-1} t_{d-1,d-1} &= t_{d+1,d-1}
\end{aligned}$$

so that $\alpha_{d-1} = 0$ and $t_{d+1,d-1} = 0$. Similarly we find that $t_{d+1,j} = 0$ for $1 \leq j \leq d$.

Next we consider when,

$$[x_k, t_{d+2}] \in F_n^c$$

(where $[x_k, w_1] = z_m$ and $[x_k, w_2] = z_l$). Thus and writing $t_{d+2} = w_2^{\alpha_2^{-1}} w_1^{\alpha_1} \underline{z}^{\underline{\gamma}}$ where $t_{d+2,d+1} = \frac{1}{\alpha_2}$ for $\alpha_2 \in \mathbb{N}$,

$$\begin{aligned}
[x_k, t_{d+2}] &= [x_k, w_2^{\alpha_2} w_1^{\alpha_1} \underline{z}^{\underline{\gamma}}] \\
&= [x_k, \underline{z}^{\underline{\gamma}}] \cdot [x_k, w_2^{\alpha_2} w_1^{\alpha_1}]^{\underline{z}^{\underline{\gamma}}} \\
&= [x_k, w_2^{\alpha_2} w_1^{\alpha_1}] \text{ as } \underline{z}^{\underline{\gamma}} \text{ is central} \\
&= [x_k, w_1^{\alpha_1}] \cdot [x_k, w_2^{\alpha_2}]^{w_1^{\alpha_1}} \\
&= z_m^{\alpha_1} z_l^{\alpha_2}.
\end{aligned}$$

For this to belong to G we need that $\alpha_1 \in \mathbb{Z}$ and $\alpha_2^{-1} \in \mathbb{Z}$. Thus we must have $\alpha_2 = 1$. At this point we can observe that as we know that the sequence $t_{d+2}, t_{d+1}, \ldots t_1$ is reduced, it follows (focusing on t_{d+2} that $0 \leq \alpha_1 < t_{d,d} = 1$. Since we now require $\alpha_1 \in \mathbb{Z}$, it must be the case that $\alpha_1 = 1$. However, we also must check that

$$w_2^{-1} t_{d+2} \in T_{d+1}$$

so we must be able to write

$$w_1^{\alpha_1} z_d^{t_{d+2,d}} z_{d-1}^{t_{d+2,d-1}} \cdots z_2^{t_{d+2,2}} z_1^{t_{d+2,1}} = t_{d+1}^{\beta_{d+1}} t_d^{\beta_d} t_{d-1}^{\beta_{d-1}} \cdots t_2^{\beta_2} t_1^{\beta_1}.$$

This also allows us to conclude that $\alpha_1 = 0$ as we must have

$$\begin{aligned}
\alpha_1 = \beta_{d+1} \cdot t_{d+1,d+1} &= \beta_{d+1} \\
\text{and} \qquad 0 \leq \alpha_1 < t_{d+1,d+1} &= 1.
\end{aligned}$$

We carry on checking in this way to discover that $t_{d+1,j} = 0$ for $d + 1 \geq j \geq 1$.

We end up with our good basis fragment having the form:

$$\begin{array}{ccccccccc}
t_1 & = & & & & & & & z_1^{t_{1,1}} \\
\vdots & & & & & & & & \\
t_d & = & & & & & z_d^{t_{d,d}} & 1\cdots & 1 \\
t_{d+1} & = & & & & w_1 & 1 & \cdots & 1 \\
\vdots & & & & & & & & \\
t_{d+l} & = & & w_l & 1 & \cdots & & & 1 \\
t_{d+l+1} & = & u_1^{t_{d+l+1,d+l+1}} & w_l^{t_{d+l+1,d+l}} & \cdots & w_1^{t_{d+l+1,d+1}} & z_d^{t_{d+l+1,d}} & & z_1^{t_{d+l+1,1}}
\end{array}$$

where $[x_i, u_j] = w_k$ for some $1 \leq i \leq n$. We now consider when

$$[x_i, t_{d+l+1}] \in F_n^c$$

For ease we write (in the obvious notation) $t_{d+l+1} = u_1^{\alpha^{-1}} \underline{w}^{\underline{\beta}} \underline{z}^{\underline{\gamma}}$ then

$$\begin{aligned} [x_i, t_{d+l+1}] &= [x_i, u_1^{\alpha^{-1}} \underline{w}^{\underline{\beta}} \underline{z}^{\underline{\gamma}}] \\ &= [x_i, \underline{z}^{\underline{\gamma}}][x_i, u_1^{\alpha^{-1}} \underline{w}^{\underline{\beta}}]^{\underline{z}^{\underline{\gamma}}} \\ &= [x_i, u_1^{\alpha^{-1}} \underline{w}^{\underline{\beta}}] \end{aligned}$$

since $\underline{z}^{\underline{\gamma}}$ is central. Now,

$$\begin{aligned} [x_i, u_1^{\alpha^{-1}} \underline{w}^{\underline{\beta}}] &= [x_i, \underline{w}^{\underline{\beta}}] \cdot [x_i, u_1^{\alpha^{-1}}]^{\underline{w}^{\underline{\beta}}} \\ &= [x_i, \underline{w}^{\underline{\beta}}] \cdot (w_k^{\alpha^{-1}})^{\underline{w}^{\underline{\beta}}} \\ &= [x_i, \underline{w}^{\underline{\beta}}] \cdot (w_k^{\alpha^{-1}}). \end{aligned}$$

As $[x_i, w_j] \in Z(F_n^c)$ for all $1 \leq j \leq l$ we have

$$[x_i, \underline{w}^{\underline{\beta}}] \cdot (w_k^{\alpha^{-1}}) = w_k^{\alpha^{-1}} \cdot \underline{z}^{\underline{\gamma}}$$

for some $\gamma \in \mathbb{Q}^n$. Hence we need $\alpha = 1$ since we know $\alpha \in \mathbb{N}$ and $\alpha^{-1} \in \mathbb{Z}$. We also need that $\underline{z}^{\underline{\gamma}} \in F_n^c$, however, the conditions that we are about to find to ensure that $u_1^{-1} t_{d+l+1} \in T_{d+1}$ will be sufficient to ensure this. Thus we need

$$u_1^{-1} u_1 w_l^{t_{d+l+1,d+l}} w_{l-1}^{t_{d+l+1,d+l-1}} \cdots z_1^{t_{d+l+1,1}} = t_{d+l}^{\alpha_{d+l}} t_{d+l-1}^{\alpha_{d+l-1}} \cdots t_{d+l}^{\alpha_{d+l-1}}.$$

We know that $t_{d+l+1}, t_{d+l}, \ldots, t_1$ is reduced so that, in particular, we must have (for t_{d+l+1}) $0 \leq t_{d+l+1,d+l} < t_{d+ld+l} = 1$. Hence as $\alpha_{d+l} \cdot t_{d+l,d+l} = t_{d+l+1,d+l}$ we have $t_{d+l+1d+l} = 0$. Similar arguments will show that $t_{d+l+1} = u_1$.

We proceed in this way; that is consider when $[x_k, t_i] \in F_n^c$ and find that the leading coefficient of t_i must be 1. Then use the arguments outlined above to see that the other coefficients are 0. Hence we find that our good basis is prescribed apart from the elements generating the centre and hence

$$\zeta_{F_n^c}^{\triangleleft,\mathrm{up}} = \zeta_{\mathbb{Z}^d}(s)$$

where $d = \operatorname{rank} Z(F_n^c)$. $\square$

We note that this is a surprising result. Lemma 4.16 gives a general result stating that if G is a $\mathcal{T}$-group with centre Z and $Z \leq_f K \leq Z^{\mathbb{Q}}$ we have $G \trianglelefteq_f \langle G, K\rangle$. However, the above result gives a converse to this in the case when G is a free nilpotent group.

Again we will do an example to demonstrate this calculation. Let $G = F_2^3 = \langle x, y | [x,y] = w, [x,w] = z_1, [y,w] = z_2, z_i \text{ central}\rangle$. If we have $T \geq G$ then T will have a good basis

$$\begin{array}{cccccccc} x^{a^{-1}} & y^b & w^c & z_1^d & z_2^e & = & t_5 \\ & y^{f^{-1}} & w^g & z_1^h & z_2^i & = & t_4 \\ & & w^{j^{-1}} & z_1^k & z_2^l & = & t_3 \\ & & & z_1^{m^{-1}} & z_2^n & = & t_2 \\ & & & & z_2^{p^{-1}} & = & t_1. \end{array}$$

We note that we assume that this is a good basis and that the sequence $t_i$is reduced using the $\sqsubset'_{\mathbb{Q}}$ order (as above). So, for example we have $0 \leq g < j^{-1}$. We begin by considering when $[y, t_3] \in G$. Now,

$$\begin{aligned} [y, t_3] &= [y, w^{j^{-1}} z_1^k z_2^l] \\ &= [y, z_1^k z_2^l] \cdot [y, w^{j^{-1}}]^{z_1^k z_2^l} \\ &= z_2^{j^{-1}} \end{aligned}$$

since z_1, z_2 are central. Thus we need $z_2^{j^{-1}} \in G$ hence $j^{-1} \in \mathbb{Z}$. Therefore we have $j \in \mathbb{N}$ so $j = 1$. We now check if $w^{-1}t_3 \in T_2$, that is if $z_1^k z_2^l = t_2^{\alpha} t_1^{\beta}$. So we need

$$\begin{aligned} \alpha m^{-1} &= k \\ \text{and} \quad l &= \alpha n + \beta p^{-1}. \end{aligned}$$

Since t_3, t_2, t_1 is reduced we know that

$$0 \leq k < m^{-1} \text{ and } 0 \leq l, n < p^{-1}.$$

Thus we must have $k = l = 0$. So our good basis becomes

$$\begin{array}{cccccccc} x^{a^{-1}} & y^b & w^c & z_1^d & z_2^e & = & t_5 \\ & y^{f^{-1}} & w^g & z_1^h & z_2^i & = & t_4 \\ & & w & 1 & 1 & = & t_3 \\ & & & z_1^{m^{-1}} & z_2^n & = & t_2 \\ & & & & z_2^{p^{-1}} & = & t_1. \end{array}$$

We now check when $[x, t_4] \in G$. Thus,

$$\begin{aligned} [x, t_4] &= [x, y^{f^{-1}} w^g z_1^h z_2^i] \\ &= [x, y^{f^{-1}} w^g] \text{ as } z_1, z_2 \text{ are central} \\ &= [x, w^g][x, y^{f^{-1}}]^{w^g} \\ &= z_1^g \cdot (w^{f^{-1}})^{w^g} \\ &= w^{f^{-1}} z_1^g \end{aligned}$$

and so we have $f = 1$ (as $f \in \mathbb{N}$ and we must have $f^{-1} \in \mathbb{Z}$) and $g = 0$ (as $g \in \mathbb{Z}$ and $0 \leq g < 1$) if $[x, t_4] \in G$. As before we check when $y^{-1} t_4 \in T_3$, thus we must check that

$$z_1^h z_2^i = t_3^\alpha t_2^\beta t_1^\gamma.$$

Clearly $\alpha = 0$, and since we know that $0 \leq h < m^{-1}$ and $0 \leq i < p^{-1}$ and have $\alpha m^{-1} = h$ and $\alpha n + \beta p^{-1} = i$ it must be the case that $h = i = 0$. Thus our good basis becomes

$$\begin{array}{cccccccc} x^{a^{-1}} & y^b & w^c & z_1^d & z_2^e & = & t_5 \\ & y & 1 & 1 & 1 & = & t_4 \\ & & w & 1 & 1 & = & t_3 \\ & & & z_1^{m^{-1}} & z_2^n & = & t_2 \\ & & & & z_2^{p^{-1}} & = & t_1. \end{array}$$

Finally we consider when $[y, t_5] \in G$.

$$\begin{aligned} [y, t_5] &= [y, x^{a^{-1}} y^b w^c z_1^d z_2^e] \\ &= [y, x^{a^{-1}} y^b w^c] \text{ as } z_1, z_2 \text{ are central} \\ &= [y, w^c] \cdot [y, x^{a^{-1}} y^b]^{w^c} \\ &= z_2^c([y, y^b][y, x^{a^{-1}}]^{y^b})^{w^c} \\ &= z_2^c(w^{a^{-1}})^{y^b w^c}. \end{aligned}$$

Now if we consider $(w^{a^{-1}})^{y^b w^c}$, we find,

$$\begin{aligned}(w^{a^{-1}})^{y^b w^c} &= (y^{-b} w^{a^{-1}} y^b \cdot 1)^{w^c} \\ &= (y^{-b} w^{a^{-1}} y^b w^{-a^{-1}} w^{a^{-1}})^{w^c} \\ &= ([y^b, w^{-a^{-1}}] w^{a^{-1}})^{w^c} \\ &= z_2^{-a^{-1}b} w^{a^{-1}}\end{aligned}$$

as z_2 is central. Hence,

$$[y, t_5] = w^{a^{-1}} \cdot z_2^{c-a^{-1}b}$$

so we need $a^{-1} \in \mathbb{Z}$ and hence $a = 1$. We now check to see if

$$x^{-1} t_5 = y^b w^c z_1^d z_2^e \in T_4.$$

Thus we try to write

$$y^b w^c z_1^d z_2^e = t_4^{\alpha} t_3^{\beta} t_2^{\gamma} t_1^{\delta}.$$

Firstly $b = \alpha \in \mathbb{Z}$ and we must have $0 \leq b < 1$ so that $\alpha = 0$. Similarly we now need that $c = \beta \in \mathbb{Z}$ so that $c = 0$ since $0 \leq c < 1$. Similar arguments will give $d = e = 0$ so that our good basis for $G \trianglelefteq T$ will look like

$$\begin{array}{cccccccc} x & 1 & 1 & 1 & 1 & = & t_5 \\ & y & 1 & 1 & 1 & = & t_4 \\ & & w & 1 & 1 & = & t_3 \\ & & & z_1^{m^{-1}} & z_2^{n} & = & t_2 \\ & & & & z_2^{p^{-1}} & = & t_1, \end{array}$$

and thus we have

$$\zeta_{F_2^3}^{\mathrm{up},\triangleleft}(s) = \zeta(s)\zeta(s-1).$$

Chapter 5

Isomorphism and Co-index Isomorphism Zeta Functions

We now introduce the isomorphism and co-isomorphism zeta functions. In §5.1 we define terminology and obtain some results for the subgroup case before considering the overgroup case for $\mathcal{H}$-groups and F_n^c. We also calculate the partial co-index zeta functions for $\mathcal{H}$-groups in an analogous way to the subgroup case.

5.1 Isomorphism Zeta Functions

We begin with the following definition.

Definition. *[Smi83, Definition 1.7], [GSS88] Let G be a finitely generated group, then the* isomorphism zeta function *of G is*

$$\zeta_G^{iso}(s) = \sum_{\substack{H \leq_f G \\ H \simeq G}} |G:H|^{-s} = \sum_{n \in \mathbb{N}} a_n n^{-s}$$

where $a_n = a_n(G) = |\{H | H \leq_f G, H \simeq G, |G:H| = n\}|$.

As always we may relax the condition that G is finitely generated provided we have that $a_n < \infty$ for all $n \in \mathbb{N}$. We now state some results for isomorphism zeta functions of $\mathcal{T}$-groups. Firstly we have

Theorem 5.1. *Let $G = C_\infty^d$ then*

$$\zeta_G^{iso}(s) = \zeta_G(s) = \zeta(s)\zeta(s-1)\cdots\zeta(s-d+1).$$

This holds because each subgroup of finite index in C_∞^d is a copy of C_∞^d.

If we define α_G^{iso} to be the abscissa of convergence of $\zeta_G^{\mathrm{iso}}(s)$ we then have,

Corollary 5.2. *If $G = C_\infty^d$ then $\alpha_G^{iso} = d$.*

Next we consider $\mathcal{H}$-groups. To prove the following result we find a good basis for an overgroup T of $G \in \mathcal{H}$ method and apply Lemma 2.10 to tell us when $T \simeq G$. Thus we have,

Theorem ([Smi83, Theorem 1.7]). *Let $G \in \mathcal{H}$. Then*

$$\zeta_G^{iso}(s) = \zeta(2s-1)\zeta(2s-2).$$

Corollary 5.3. *If $G = \mathcal{H}$ then $\alpha_G^{iso} = 2$.*

Finally, we state the following result (again in [Smi83]) and we will seek the co-isomorphism analogue

Theorem ([Smi83, Theorem 1.9]). *Let $G = F_n^c$ and*

$$\lambda_c = \sum_{i=1}^{c} h(\gamma_i(G))$$

then

$$\zeta_G^{iso}(s) = \zeta_{C_\infty^n}\left((s-n)\frac{\lambda_c}{n} + n\right)$$

Corollary 5.4. *If $G = F_n^c$ then $\alpha_G^{iso} = n$.*

We now consider the co-isomorphism case.

5.2 Co-isomorphism Zeta Functions

Definition. *Let G be a finitely generated torsion free nilpotent group. We define the* co-isomorphism zeta function *of G to be*

$$\zeta_G^{up,\ iso}(s) = \sum_{\substack{G \le T \le G^{\mathbb{Q}} \\ |T:G|<\infty \\ G \simeq T}} |T : G|^{-s}.$$

Our aim is to calculate, as with the subgroup case, this zeta function for F_n^c. Firstly however we consider the cases $G = C_\infty^n$ and $G \in \mathcal{H}$.

Proposition 5.5. *If $G = C_\infty^d$ then*

$$\zeta_G^{up,\ iso}(s) = \zeta_G(s).$$

Corollary 5.6. $\alpha_{C_\infty^d}^{up,iso} = d$.

(Where $\alpha^{\mathrm{up,iso}}$ is the abscissa of convergence of $\zeta_G^{\mathrm{up,\ iso}}$.)

As before this result is immediate. We now perform a simple calculation (Lemma 5.8), which will show us when an overgroup of a $\mathcal{H}$-group G is isomorphic to G. This will then enable us to easily prove,

Theorem 5.7. *Let $G \in \mathcal{H}$, that is let $G = \langle x, y | [x, y] = z^t, [x, z] = [y, z] = 1\rangle$ for some $t \in \mathbb{N}$. Then*

$$\zeta_G^{up,\ iso}(s) = \zeta(2s-1)\zeta(2s-2).$$

Lemma 5.8. *Let $G \in \mathcal{H}$, that is let $G = \langle x, y | [x, y] = z^t, [x, z] = [y, z] = 1\rangle$ for some fixed $t \in \mathbb{N}$. If $G \le_f T \le G^{\mathbb{Q}}$, then T will have good basis*

$$\begin{array}{ccc} x^{a^{-1}} & y^b & z^c \\ & y^{d^{-1}} & z^e \\ & & z^{f^{-1}}. \end{array}$$

Let $H_r \in \mathcal{H}$, where $H_r = \langle w, u | [w, u] = v^r, [w, v] = [u, v] = 1\rangle$. In this situation we have $T \simeq H_r$ if $ft = rad$.

Proof. We define a map $\theta : H_r \to T$ as follows,

$$\begin{array}{ccccc} w & \mapsto & x^{a^{-1}} & y^b & z^c \\ u & \mapsto & & y^{d^{-1}} & z^e \\ v & \mapsto & & & z^{f^{-1}} \end{array}$$

and extend to H_r in the obvious manner. As this is a map between two bases, if θ is a homomorphism then we will have $H_r \simeq T$. This is what we check, *i.e.*,

$$([w,v])\theta = (v^r)\theta.$$

Now,

$$\begin{aligned} ([w,u])\theta &= (w^{-1}v^{-1}wv)\theta \\ &= (x^{a^{-1}}y^bz^c)^{-1}(y^{d^{-1}}z^e)^{-1}(x^{a^{-1}}y^bz^c)(y^{d^{-1}}z^e) \\ &= (x^{-a^{-1}}y^{-b}z^{-c-ta^{-1}b})(y^{-d^{-1}}z^{-e})(x^{a^{-1}}y^bz^c)(y^{d^{-1}}z^e) \end{aligned}$$

(Recalling that in H_r we have

$$\begin{array}{rcl} \text{Multiplication} & w^au^bv^c \cdot w^du^ev^f = & w^{a+d}u^{b+e}v^{c+e-rbd} \\ \text{Inversion} & (w^au^bv^c)^{-1} = & w^{-a}u^{-b}v^{-c-rab} \\ \text{Exponentiation} & (w^au^bv^c)^n = & w^{na}u^{nb}v^{nc-r\binom{n}{2}ab} \end{array}$$

where $\binom{n}{2} = \frac{n!}{(n-2)!2!}$).

Now, we have

$$\begin{aligned} ([w,u])\theta &= (x^{-a^{-1}}y^{-b}z^{-c-ta^{-1}b})(y^{-d^{-1}}z^{-e})(x^{a^{-1}}y^bz^c)(y^{d^{-1}}z^e) \\ &= (x^{-a^{-1}}y^{-(b+d^{-1})}z^{-c-ta^{-1}b-e})(x^{a^{-1}}y^{(b+d^{-1})}z^{c+e})) \\ &= (x^{-a^{-1}}y^{-(b+d^{-1})}z^{-c-ta^{-1}b-e})(y^{(b+d^{-1})}x^{a^{-1}}z^{ta^{-1}(b+d^{-1})}z^{c+e})) \\ &= z^{-ta^{-1}b+ta^{-1}(b+d^{-1})} \end{aligned}$$

Thus we have,

$$([w,u])\theta = z^{(ta^{-1}d^{-1})}$$

and require that

$$(v^r)\theta = ([w,u])\theta = z^{(ta^{-1}d^{-1})}.$$

Hence we must have,

$$(v^r)\theta = (z^{f^{-1}})^r = z^{(ta^{-1}d^{-1})}.$$

In other words we must have $ft = rad$if $T \simeq H_r$. $\square$

We are now in a position to to calculate $\zeta_G^{\text{up,iso}}(s)$ for $G \in \mathcal{H}$ and hence prove Theorem 5.7.

Proof of Theorem 5.7. Recall we are in the following situation. We have $G \in \mathcal{H}$, with $G = \langle x, y | [x, y] = z^t, [x, z] = [y, z] = 1\rangle$ for some $t \in \mathbb{N}$. Let T be an overgroup of finite co-index G, that is $G \leq_f T \leq G^{\mathbb{Q}}$. Thus we have the following good basis for T,

$$\begin{array}{ccc} x^{a^{-1}} & y^b & z^c \\ & y^{d^{-1}} & z^e \\ & & z^{f^{-1}}, \end{array}$$

with $a, d, f \in \mathbb{N}$ and $b, c, e \in \mathbb{Q}$. We have previously shown in Theorem 3.11 that if this is a good basis we must have

- $f = \lambda ad$ for some $\lambda \in \mathbb{Z}$,
- $b = \frac{l}{ad}$ with $0 < l \leq a$,
- $e = \frac{m}{df}$ with $0 < m \leq d$,
- a distinct choices for c.

We have also just shown in Lemma 5.8 that if $T \simeq G$ we must have $tad = tf$ or in other words, $ad = f$, which then replaces the first condition above. Hence we have,

$$\begin{aligned} \zeta_G^{\text{up, iso}}(s) &= \sum_{\substack{a,d,f \in \mathbb{N} \\ ad=f}} (adf)^{-s} a^2 d \\ &= \sum_{a,d \in \mathbb{N}} (ad)^{-2s} a^2 d \\ &= \zeta(2s-1)\zeta(2s-2). \end{aligned}$$

$\square$

Corollary 5.9. *Let $H \in \mathcal{H}$ then $\alpha_H^{up,iso} = \frac{3}{2}$.*

5.3 Partial Co-index zeta functions

In a similar fashion to the subgroup case we make the following definition of a partial co-index zeta function.

Definition. *Let $H_r, H_t \in \mathcal{H}$ for some $r, t \in \mathbb{N}$. Define*

$$\phi_{H_t}^{up,H_r}(s) = \sum_{\substack{H_t \leq_f T \leq (H_t)^{\mathbb{Q}} \\ T \simeq H_r}} |T : H_t|^{-s}.$$

We also set

$$\phi_{H_t}^{p,up,H_r}(s) = \sum_{\substack{H_t \leq_f T \leq (H_t)^{\mathbb{Q}} \\ T \simeq H_r \\ |T:H_t| \text{ a } p \text{ power}}} |T : H_t|^{-s}.$$

Our aim is to calculate $\phi_{H_t}^{\mathrm{up},H_r}(s)$. We will firstly consider the case when $t = 1$, thus,

Proposition 5.10. *Let $H = F_2^{\mathfrak{H}}$ and $H_r \in \mathcal{H}$ for some $r \in \mathbb{N}$. Then*

$$\phi_H^{up,H_r}(s) = r^{-s}\zeta(2s-1)\zeta(2s-2).$$

Proof. We know from Theorem 3.9 that an overgroup T of H such that $H \leq_f T \leq H^{\mathbb{Q}}$ will have a good basis

$$\begin{array}{ccc} x^{\alpha^{-1}} & y^{\beta} & z^{\gamma} \\ & y^{\delta^{-1}} & z^{\varepsilon} \\ & & z^{\lambda^{-1}} \end{array}$$

with $\alpha, \delta, \lambda \in \mathbb{N}$ and with α choices of valid β and γ and δ valid choices of ε. Further if we want $T \simeq H$ Lemma 5.8 means that we require $\lambda = r\alpha\delta$. Hence we have

$$\begin{aligned}
\phi_H^{\mathrm{up},H_r}(s) &= \sum_{\alpha\in\mathbb{N}}\sum_{\delta\in\mathbb{N}}\sum_{\substack{\lambda\in\mathbb{N}\\ \lambda = r\alpha\delta}} (\alpha\delta\lambda)^{-s}\alpha^2\delta \\
&= \sum_{\alpha\in\mathbb{N}}\sum_{\delta\in\mathbb{N}} r^{-s}(\alpha\delta)^{-2s}\alpha^2\delta \\
&= r^{-s}\zeta(2s-1)\zeta(2s-2)
\end{aligned}$$

as required. □

In the case when $t > 1$ things are more complicated.

Proposition 5.11. *Let $H_t, H_r \in \mathcal{H}$. If $\frac{r}{t} \in \mathbb{N}$ then*

$$\phi_{H_t}^{\mathrm{up},H_r}(s) = \frac{r^{-s}}{t^{-s}}\zeta(2s-1)\zeta(2s-2).$$

Otherwise let θ be the p^{th} part of t and ρ be the p^{th} part of r then if $r - t \geq 0$ we have

$$\phi_{H_t}^{p,\mathrm{up},H_r}(s) = p^{\rho-\theta}\zeta^p(2s-1)\zeta^p(2s-2),$$

otherwise we have

$$\phi_{H_t}^{p,\mathrm{up},H_r}(s) = \frac{p^{(\rho-\theta)(1-3s)}}{(1-p)}\left(\zeta^p(2s-1) - p^{(\rho-\theta+1)}\zeta^p(2s-2)\right).$$

Proof. As above we know that a good basis for a finite co-index overgroup T of H_t is given by

$$\begin{array}{ccc}
x^{\alpha^{-1}} & y^{\beta} & z^{\gamma} \\
 & y^{\delta^{-1}} & z^{\varepsilon} \\
 & & z^{\lambda^{-1}}
\end{array}$$

where $\alpha, \delta, \lambda \in \mathbb{N}$ and we have α choices for valid β and γ and δ choices for ε (see Theorem 3.11). Further to these we need that $\lambda t = r\alpha\delta$ if we are to have $T \simeq H_r$. Thus, if $\frac{r}{t} \in \mathbb{N}$ we can proceed to set $\lambda = \frac{r}{t}\alpha\delta$ and we have

$$\phi_{H_t}^{\mathrm{up},H_r}(s) = \frac{r^{-s}}{t^{-s}}\zeta(2s-1)\zeta(2s-2).$$

(This calculation is similar to Proposition 5.10.) If $\frac{r}{t} \notin \mathbb{N}$ then the calculation is more problematic and so move to considering the case of a single prime $p \in \mathcal{P}$. Thus let the p^{th} part of $\alpha, \delta, \lambda, r$ and t be p^a, p^d, p^l, p^ρ and p^θ respectively. We then have

$$\begin{aligned}
\phi_{H_t}^{p,\mathrm{up},H_r}(s) &= \sum_{\substack{a\in\mathbb{N} \\ l+\theta=\rho+a+d}} \sum_{d\in\mathbb{N}} \sum_{l\in\mathbb{N}} (p^a p^d p^l)^{-s} p^{2a} p^d \\
&= \sum_{\substack{a\in\mathbb{N} \\ l=(\rho-\theta)+a+d}} \sum_{d\in\mathbb{N}} (p^{2a} p^{2d} p^{(\rho-\theta)})^{-s} p^{2a} p^d.
\end{aligned}$$

If $\rho - \theta \geq 0$ then the above sum can easily be seen to be

$$\phi_{H_t}^{p,\mathrm{up},H_r}(s) = p^{\rho-\theta}\zeta^p(2s-1)\zeta^p(2s-2).$$

If $\rho - \theta < 0$ then the calculation is less clear, but eventually we can see that

$$\phi_{H_t}^{p,\mathrm{up},H_r}(s) = \frac{p^{(\rho-\theta)(1-3s)}}{(1-p)} \left(\zeta^p(2s-1) - p^{(\rho-\theta+1)}\zeta^p(2s-2)\right).$$

□

5.4 The Co-isomorphism Zeta Function for F_n^c

We now aim to calculate $\zeta_{F_n^c}^{\mathrm{up,iso}}(s)$ where as always F_n^c is the free nilpotent group of class c on n generators. We calculate this zeta function using a sequence of lemmas.

Lemma 5.12. *Let $G = F_n^c = \langle x_n, x_{n-1} \ldots, x_1\rangle$ and $G \leq_f T \leq G^{\mathbb{Q}}$ with T given by a good basis $T = \langle t_m, t_{m-1}, \ldots, t_1\rangle$ (here $m = h(G)$). If $T \simeq G$ then T is generated by the first n elements of the good basis.*

Note that this is just a generalisation of Lemma 5.8.

Proof. Any isomorphism $\theta : G \to T$ will send generators to generators. As G is free in its class we will have that T is generated by $t_m, \ldots t_1$ as required. □

Thus if an overgroup T of F_n^c is isomorphic to F_n^c we must have that the good basis for T is determined save for the first n basis elements – the others are then appropriate commutators of these generators. This is the same as in standard (subgroup) isomorphism zeta function case. Hence, if we know the degree of choice we have for each of these n basis elements, we can determine the co-isomorphism zeta function for F_n^c. Next, we determine the co-index of such an overgroup.

Lemma 5.13. *Let $F_n^c \leq_f T$ and let T have a good basis as follows,*

$$\begin{array}{rcllllll} t_m & = & w_m^{t_{m,m}} & w_{m-1}^{t_{m,m-1}} & & & \cdots & w_1^{t_{m,1}} \\ t_{m-1} & = & & w_{m-1}^{t_{m-1,m-1}} & w_{m-2}^{t_{m-1,m-2}} \cdots & & & w_1^{t_{m,1}} \\ \vdots & & & & & & & \\ t_{m-n} & = & & & w_{m-n}^{t_{m-n,m-n}} & w_{m-n-1}^{t_{m-n,m-n-1}} & \cdots & w_1^{t_{m-n,1}} \end{array}$$

and $t_{m-n-1} = [t_m, t_{m-1}], \ldots$ where the $w_m = x_n, w_{m-1} = x_{n-1} \ldots w_{m-n} = x_1$ and then $w_{m-n-1} = [x_n, x_{n-1}], w_{m-n-1} = [x_n, x_{n-2}], \ldots$. If $T \simeq F_n^c$ we have

$$|T : F_n^c| = \prod_{i=m-n}^{m} \left(\frac{\lambda_c}{n} t_{i,i} \right)$$

where $\lambda_c = \sum_{i=1}^{c} h(\gamma_i(F_n^c))$.

Proof. This formula counts the occurrences of the t_i in each set of basic commutators of weight j with $1 \leq j \leq c$ (see [Hal79, section 5]).This is independent of our choice of basic commutators, and to see this let

$$T^k = \langle t_m, t_{m-1}, \ldots t_{m-k}^2, \ldots, t_{m-n} \rangle.$$

and examine $|\gamma_j(T) : \gamma_j(T^k)\gamma_{j+1}(T)|$. If this is independent of k we are done.

We observe that T is free nilpotent of class c on $T_m, \ldots t_{m-n}$ (as $T \simeq F_n^c$). Now, it is easy to see that $|T : T^k T'|$ is finite which means that $|T : T^k|$ is finite. We may extend a map $\theta : T \to T^k$ which sends $t_{m-k} \mapsto t_{m-k}^2$ and $t_i \mapsto t_i$ to an epimorphism. Considering the Hirsch lengths of the groups, we find that the kernel will be finite. Hence (as T is torsion free) we see that θ is an isomorphism.

Any permutation of the generators of T induces an isomorphism of T. If the $(m-k)^{th}$ generator gets sent to the $(m-j)^{th}$ generator, say,then T^k will be map bijectively to T^j and $|T : T^k| = T : T^j|$. This ensures that $|\gamma_j(T) : \gamma_j(T^k)\gamma_{j+1}(T)|$n is independent of k. Hence the number of occurrences of t_{m-k} must be independent of the choice of basic sequence and so is

$$\frac{h\left(\frac{\gamma_j(T)}{\gamma_{j+1}(T)}\right)}{n}.$$

□

It remains to see the number of choices we have for the elements $t_m, t_{m-1} \dots, t_{m-n}$ of our good basis (and thus the whole good basis by Lemma 5.12 when $|T : F_c^n|$ is fixed and this is the content of the next lemma.

Lemma 5.14. *Let $F_n^c \leq_f T$ and let T have a good basis,*

$$\begin{array}{rcllllll}
t_m & = & w_m^{t_{m,m}} & w_{m-n}^{t_{m,m-n}} & & \cdots & & w_1^{t_{m,1}} \\
t_{m-1} & = & & w_{m-1}^{t_{m-1,m-1}} & w_{m-2}^{t_{m-1,m-2}} \cdots & & & w_1^{t_{m,1}} \\
\vdots & & & & & & & \\
t_{m-n} & = & & & w_{m-n}^{t_{m-n,m-n}} & w_{m-n-1}^{t_{m-n,m-n-1}} & \cdots & w_1^{t_{m-n,1}}
\end{array}$$

and $t_{m-n-1} = [t_m, t_{m-1}], \dots$ where the w_i are as in Lemma 5.12. We then have $t_{i,i}^{-1}$ choices for $t_{i,j}m - n \leq i \leq m$ and $1 \leq j \leq i$.

Proof. Since $t_m, t_{m-1}, \cdots t_1$ are a good basis we should have

1. $t_m, t_{m-1}, \cdots t_1$ reduced
2. $T_j = \{t_j^{\beta_j} t_{j-1}^{\beta j-1} \cdots t_1^{\beta_1} | \beta \in \mathbb{Z}\}$ a group
3. $t_{i,i} = \frac{1}{\alpha}$ for $\alpha \in \mathbb{N}$
4. $w_m^{-1} t_i^{\alpha_i} \in T_{i-1}$.

We observe that (2) and (3) are clear. We note the (1) is problematic as t_k for $1 \leq k \leq m-n$ is fixed so so we only insist that $t_m, t_{m-1}, \dots, t_{m-n}$ are reduced (this causes no problems as the other good basis elements are fixed). We thus check (4).

We firstly look at t_{m-n} so we must have

$$w_{m-n}^{-1} t_{m-n}^{\alpha} = t_{m-n-1}^{\beta_{m-n-1}} t_{m-n-2}^{\beta_{m-n-2}} \cdots t_1^{\beta_1}$$

where $t_{m-n,m-n} = \frac{1}{\alpha}$ for some $\beta_i \in \mathbb{Z}$. Now

$$w_{m-n}^{-1} t_{m-n}^{\alpha} = w_{m-n-1}^{\alpha t_{m-n,m-n-1}+\gamma_{m-n-1}} w_{m-n-2}^{\alpha t_{m-n,m-n-2}+\gamma_{m-n-2}} \cdots w_1^{\alpha t_{m-n,1}+\gamma_1}$$

for some $\gamma_j \in \mathbb{Q}$. So we need

$$\alpha t_{m-n,m-n-1} + \gamma_{m-n-1} = \beta_{m-n-1} t_{m-n-1,m-n-1}$$
$$0 < t_{m-n,m-n-1} \leq t_{m-n-1,m-n-1}$$

If, for simplicity, we write $t_{m-n-1,m-n-1} = \frac{1}{\delta}$ we have,

$$t_{m-n,m-n-1} = \frac{1}{\alpha}\left(\frac{\beta_{m-n-1}}{\delta} - \gamma_{m-n-1}\right).$$

So we need,

$$0 < \frac{1}{\alpha}\left(\frac{\beta_{m-n-1}}{\delta} - \gamma_{m-n-1}\right) \leq \frac{1}{\delta}$$

and hence we have

$$0 < \beta_{m-n-1} - \gamma_{m-n-1}\delta \leq \alpha.$$

As $\beta_{m-n-1} \in \mathbb{Z}$, the number of choices for β_{m-n-1} and hence for $t_{m-n,m-n-1}$ is $|(\mathbb{Z} - d) \cap (0, \alpha]|$ where $d = \gamma_{m-n-1}\delta$. Thus we have α choices for $t_{m-n,m-n-1}$.

We now consider the choices for $t_{m-n,m-n-2}$; thus we have

$$\alpha t_{m-n,m-n-2} + \gamma_{m-n-2} = \beta_{m-n-1} t_{m-n-1,m-n-2} + \delta_{m-n-1} + \beta_{m-n-2} t_{m-n-2,m-n-2}$$

As $\beta_{m-n-1} t_{m-n-1,m-n-2} + \delta_{m-n-1}$ is fixed we will write this as s. We also need

$$0 < t_{m-n,m-n-2} \leq t_{m-n-2,m-n-2}$$

Thus we write,

$$t_{m-n,m-n-2} = \frac{1}{\alpha}\left(s - \gamma_{m-n-2} = \frac{\beta_{m-n-1}}{\delta}\right)$$

where $t_{m-n-2,m-n-2} = \frac{1}{\delta}$. Hence we need that

$$0 < \frac{1}{\alpha}\left(s - \gamma_{m-n-2} + \frac{\beta_{m-n-2}}{\delta}\right) \leq \frac{1}{\delta}$$

or

$$0 < r\delta + \beta_{m-n-2} \leq \alpha$$

where $r = s - \gamma_{m-n-2}$ is fixed, so again we have α choices for $t_{m-n,m-n-2}$.

We proceed in this way along this basis element to find at each step we have α choices for $t_{m-n,i}$, $1 \leq i \leq m-n-1$. If we consider each row in turn, in this manner, we arrive at the stated conclusion. $\square$

We are now in a position to prove

Theorem 5.15. *Let F_n^c be the free nilpotent group of class c on n generators. Then*
$$\zeta_{F_n^c}^{up,iso}(s) = \zeta_{\mathbb{Z}^n}(\frac{\lambda_c}{n}s - (\lambda_c - 1)).$$

Proof. Let T be an overgroup of F_n^c of finite co-index such that $T \simeq F_n^c$ with good basis $t_m, t_{m-1}, \ldots, t_1$. Lemma 5.13 tells us that

$$|T : F_n^c| = \prod_{i=m-n}^{m} (\lambda_c t_{i,i}^{-1})$$

and Lemma 5.14 tells is that we have

$$\sum_{i=m-n}^{m} (t_{i,i}^{-1})^{\lambda_c - 2i-(m-n)}$$

choices for our good basis.

Hence if we let $t_{i,i} = \frac{1}{\alpha_i}$ we have

$$\begin{aligned}
\zeta_{F_n^c}^{\mathrm{up,iso}}(s) &= \sum_{\alpha_m \in \mathbb{N}} \sum_{\alpha_{m-1} \in \mathbb{N}} \cdots \sum_{\alpha_{m-n} \in \mathbb{N}} \left(\prod_{i=m-n}^{m} \frac{\lambda_c}{n} \alpha_i \right)^{-s} \cdot \alpha_m^{\lambda_c - 1} \alpha_{m-1}^{\lambda_c - 2} \cdots \alpha_{m-n}^{\lambda_c - n} \\
&= \zeta(\frac{\lambda_c}{n}s - (\lambda_c - 1))\zeta(\frac{\lambda_c}{n}s - (\lambda_c - 2)) \cdots \zeta(\frac{\lambda_c}{n}s - (\lambda_c - n)) \\
&= \zeta_{\mathbb{Z}^n}(\frac{\lambda_c}{n}s - (\lambda_c - 1)).
\end{aligned}$$

$\square$

Corollary 5.16. *Let F_n^c be the free nilpotent group on n generators. Then $\alpha^{up,iso} = n$*

5.5 Bounds for Co-index Zeta functions

We recall from [Smi83] the following partial ordering on Dirichlet series with non-negative real coefficients.

Definition. *If*

$$f(s) = \sum_{n\in\mathbb{N}} a_n n^{-s} \text{and } g(s) = \sum_{n\in\mathbb{N}} b_n n^{-s}$$

with $a_i, b_i \in \mathbb{R}$ and $a_i, b_i \geq 0, \forall i \in \mathbb{N}$ then if $a_i \leq b_i$ for all $i \in \mathbb{N}$ we write $f(s) \lessdot g(s)$.

We also have,

Lemma 5.17. *[Smi83, Lemma 1.1] If $f(s) \lessdot g(s)$ then $\alpha_f \leq \alpha_g$*

Proof. The dominated convergence theorem. ☐

With this in mind the above theorem allows us to give the following (obvious) lower bound for $\zeta^{\mathrm{up}}_{F^c_n}(s)$.

Proposition 5.18. *Let F^c_n be the free nilpotent group of class c on n generators. Then*

$$\zeta^{up,iso}_{F^c_n}(s) \lessdot \zeta^{up}_{F^c_n}(s)$$

so that

$$\alpha^{up,iso}_{F^c_n} \leq \alpha^{up}_{F^c_n}.$$

We now make some observations about overgroups of F^c_n using the information contained within the proof of the formula for the co-isomorphism zeta function and the proof of the co-normal zeta function. Firstly we have

Proposition 5.19. *Let F^c_n be the free nilpotent group of class c on n generators. If T is a proper overgroup of F^c_n of finite co-index such that $F^c_n \trianglelefteq T$ then $T \not\cong F^c_n$.*

(Note by proper we mean of co-index larger that 1.)

Proof. Let T be a proper overgroup of F^c_n with good basis $t_m, t_{m-1}, \ldots, t_1$. Let $d = \mathrm{rank}(Z(F^c_n))$. If $F^c_n \trianglelefteq T$ then we have that $t_m, t_{m-1}, \ldots t_{d+1}$ are fixed, with $t_j = w_j$ for $m \geq j \geq d+1$ and we are free to select $t_d, t_{d-1}, \ldots, t_1$ subject to the constraint that the t_i remain a good basis. However, if $T \simeq F_n^c$ then we are not free to choose $t_d, t_{d-1}, \ldots, t_1$ and they are fixed - determined by first n basis elements. Hence $T \not\simeq F_n^c$. □

This proposition allows us to prove the following lower bound for $\zeta^{\text{up}}_{F_n^c}(s)$:

Theorem 5.20. *Let F_n^c be the free nilpotent group of class c on n generators. We then have*

$$\zeta^{up,\triangleleft}_{F_n^c}(s)\zeta^{up,iso}_{F_n^c}(s) \prec \zeta^{up}_{F_n^c}(s).$$

Proof. Proposition 5.19 gives us the fact that we do not count anything twice. Also if $F_n^c \leq_f T, R$ with $F_n^c \trianglelefteq R$ and $T \simeq F_n^c$ we have $TR = \langle T, R\rangle \geq_f F_n^c$. □

Observing that we have

$$\zeta^{\text{up,iso}}_{F_2^2}(s) = \zeta^{\text{up},\triangleleft}_{F_2^2}(s)\zeta^{\text{up}}_{F_2^2}(s)$$

the above is a sharp lower bound. It also leads to the following conjecture,

Conjecture 5.21. *Let F_n^2 be the free nilpotent group of class 2 on n generators. Then*

$$\zeta^{up,iso}_{F_n^2}(s) = \zeta^{up,\triangleleft}_{F_n^2}(s)\zeta^{up}_{F_n^2}(s)$$

Heuristically if we have $F_n^c \leq T$ with good basis $t_m, t_{m-1}, \ldots, t_1$ we know that the co-isomorphism function gives us information about how we may select basis elements $t_m, t_{m-1}, \ldots t_{d+1}$ (note that $m = d + n$ where $d = \text{rank}(Z(F_n^2))$) and these then determine the rest of the basis. Similarly if $F_n^2 \trianglelefteq T$ we have that $t_m, t_{m-1}, \ldots t_{d+1}$ are fixed and $t_{d+1}, t_{d+1}, \ldots t_1$ can be chosen. This gives us the above result. We note that if $c > 2$ then this argument does not hold water since we do not have information on the choices of basis elements $t_{m-n}, t_{m-n-1}, \ldots t_{d+1}$.

We also observe the following, which we try to extend to the co-index case.

Lemma 5.22. *[Smi83, Lemma 1.2] If G, L are finitely generated groups and $\varphi : G \to L$ is an epimorphism then*

$$\zeta_L(s) \prec \zeta_G(s)$$

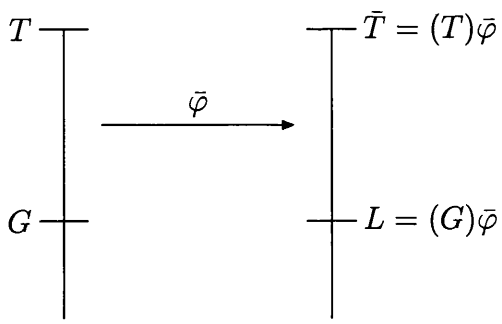


Figure 5.1: The mapping $\bar{\varphi}$

Proof.

$$\zeta_L(s) = \sum_{H \leq_f L} |L : H|^s = \sum_{\substack{H \leq_f G \\ \operatorname{Ker}\varphi \leq H}} |G : H|^s < \sum_{H \leq_f G} |G : H|^s = \zeta_G(s)$$

□

This then allows the following bound,

Corollary 5.23. *Let G and L be as in Lemma 5.22 then $\alpha_L \leq \alpha_G$.*

We now try to extend this to the upwards case. Thus let $G, L \in \mathcal{T}$ and assume we have an epimorphism

$$\varphi : G \to L$$

We now form the Mal'cev completion $G^{\mathbb{Q}}$ of G and $L^{\mathbb{Q}}$ of L. As we wish to compare the number of overgroups of G and L we extend this map φ to

$$\varphi^{\mathbb{Q}} : G^{\mathbb{Q}} \to L^{\mathbb{Q}}$$

which is the unique extension of φ (and so an epimorphism) which we know we can form by properties of Mal'cev completions. Thus given an overgroup of T of finite co-index we wish to know $|(T)\varphi^{\mathbb{Q}} : (G)\varphi^{\mathbb{Q}}| = |(T)\varphi^{\mathbb{Q}} : L|$. We restrict our attention to the map

$$\bar{\varphi} : T \to (T)\varphi^{\mathbb{Q}}$$

and for ease set $(T)\varphi^{\mathbb{Q}} = \overline{T}$. Thus we have the situation as in figure 5.1.

Now, as $\bar{\varphi}$ will preserve indices of groups above the kernel we have

$$|\bar{T} : L| = |T/K : GK/K|$$

(noting that as $K \trianglelefteq T$ we have $GK \leq T$ and that $(G)\bar{\varphi} = L$). If we now apply the third isomorphism theorem we see that

$$|T/K : GK/K| = |T : GK|.$$

Hence if $\operatorname{Ker}\bar{\varphi} \leq G$ we can use the above estimates. However in most cases this will not be true and so equivalent overgroups of L will have smaller co-index. Thus in particular there is no co-index generalisation of Lemma 5.22.

Chapter 6

Sideways Zeta functions

Let G be a finitely generated torsion free nilpotent group. We have seen that we can embed G into its Mal'cev completion, $G^{\mathbb{Q}}$, and in some cases we have calculated the co-index function for G in $G^{\mathbb{Q}}$, where

$$\zeta_G^{\mathrm{up}}(s) = \sum_{G \leq_f T \leq G^{\mathbb{Q}}} |T : G|^{-s}.$$

We now wish to consider the notion of groups to the "side of" a such a group G. Thus we make the following definition.

Definition. *Let G be a $\mathcal{T}$-group, with Mal'cev completion $G^{\mathbb{Q}}$. We say a group $K \leq G^{\mathbb{Q}}$ has* sideways distance n *from G if there is a group T such that*

1. *$G \leq_f T$ and $|T : G| = n$*
2. *$K \leq_f T$ and $|T : K| = n$*
3. *$\langle G, K \rangle = T$.*

We denote this by $|G : K|_{side} = n$.

We note that this is not the only way to define this idea of being "to the side of" a group and we will consider some alternatives later. Using this definition we now define the sideways zeta function,

Definition. *Let G be a finitely generated torsion free nilpotent group. The* sideways zeta function *of G in $G^{\mathbb{Q}}$ is*

$$\zeta_G^{side}(s) = \sum_{\substack{K \le G^{\mathbb{Q}} \\ |G:K|_{side} < \infty}} |G : K|_{side}^{-s}.$$

We will now attempt to calculate this in some cases. Firstly let $G = C_\infty^d$, the free abelian group with d generators. We know that $G^{\mathbb{Q}} \simeq \mathbb{Q}^d$ and recall that we have already calculated

$$\begin{aligned} \zeta_G(s) &= \zeta(s)\zeta(s-1)\cdots\zeta(s-d+1) = \sum_{n\in\mathbb{N}} a_n n^{-s} \\ \zeta_G^{\text{up}}(s) &= \zeta(s)\zeta(s-1)\cdots\zeta(s-d+1) = \sum_{n\in\mathbb{N}} b_n n^{-s} \end{aligned}$$

(we denote these functions using a_n and b_n for clarity of exposition as will become clear in a moment; obviously in this case $a_n = b_n$). Let

$$\zeta_G^{\text{side}}(s) = \sum_{n\in\mathbb{N}} t_n n^{-s}$$

and so clearly t_n is the number of groups to the side of G. Working with the free abelian groups we are in a favourable position as if we have $H \le_f G \le_f T$ then $G \simeq T$ and $G \simeq H$.

We will calculate t_n by counting all the groups $K \le G^{\mathbb{Q}}$ such that $K \le_f T$ and $|T : G| = |T : K| = n$, and not worrying if we over-count. This of course is another possible definition of a sideways distance away from G (without the over-counting). There are $b_n a_n$ such groups (there are b_n choices for overgroups T of co-index n and for each $T \simeq G$ we have a_n possible subgroups of index n).

We now count these groups in a different way. Let $K \le G^{\mathbb{Q}}$ be such that $|G : K|_{\text{side}} = k$ where $k|n$. Then we have that $\langle G, K\rangle \le T$ for some overgroup $G \le_f T \le G^{\mathbb{Q}}$ with $|T : G| = n$. Thus we have $|T : \langle G, K\rangle| = \frac{n}{k}$ and we are in the situation as in Figure 6.1. There are t_k choices of such K and then $b_{n/k}$ possible overgroups T of $\langle G, K\rangle$ and hence there are

$$\sum_{k|n} t_k b_{n/k}$$

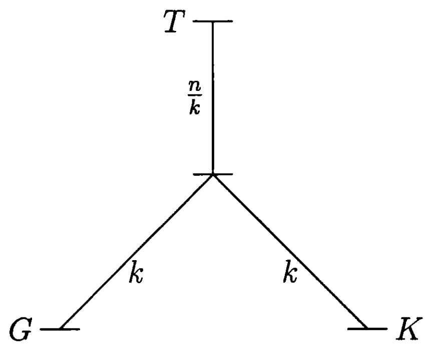


Figure 6.1: A group to the side of G

groups that we wish to count. Thus we have

$$a_n b_n = \sum_{k|n} t_k b_{n/k}$$

and so we have a formula for t_n. If we take the above, multiply both sides by n^{-s} and sum over all $n \in \mathbb{N}$ we find that

$$\sum_{n\in\mathbb{N}} a_n b_n n^{-s} = \sum_{n\in\mathbb{N}} \sum_{k|n} t_k b_{n/k} n^{-s}. \tag{6.1}$$

Recalling that if $f(s) = \sum_{n\in\mathbb{N}} f_n n^{-s}$ and $g(s) = \sum_{n\in\mathbb{N}} g_n n^{-s}$, then their product is

$$f(s)g(s) = \sum_{n\in\mathbb{N}} \sum_{k|n} f_n g_{n/k} n^{-s}$$

we see that (6.1) becomes,

$$\sum_{n\in\mathbb{N}} a_n b_n n^{-s} = \left(\sum_{n\in\mathbb{N}} t_n n^{-s}\right) \left(\sum_{n\in\mathbb{N}} b_n n^{-s}\right).$$

Defining

$$\eta_G(s) = \sum_{n\in\mathbb{N}} a_n b_n n^{-s}$$

we see that the above becomes

$$\eta_G(s) = \zeta_G^{\mathrm{side}}(s) \zeta_G^{\mathrm{up}}(s)$$

and we have shown that

Theorem 6.1. *Let $G = C_\infty^d$. Then we have*

$$\eta_G(s) = \zeta_G^{side}(s)\zeta_G^{up}(s)$$

where

$$\eta_G(s) = \sum_{n\in\mathbb{N}} a_n b_n n^{-s}$$

with a_n the number of subgroups of G of index n and b_n the number of overgroups of G of co-index n.

Unfortunately we are not always in this happy situation. In most cases we will not have that if $H \leq_f G \leq_f T \leq G^{\mathbb{Q}}$ then $G \simeq T$ and $G \simeq H$. As an approximation to this situation we can define the sideways isomorphism zeta function,

Definition. *Let G be a finitely generated torsion free nilpotent group with Mal'cev completion $G^{\mathbb{Q}}$. The* sideways isomorphism zeta function *of G in $G^{\mathbb{Q}}$ is*

$$\zeta_G^{side,\ iso}(s) = \sum_{\substack{K\leq G^{\mathbb{Q}} \\ K\simeq G \\ |G:K|_{side}<\infty \\ \langle G,K\rangle\simeq G}} |G:K|_{side}^{-s}.$$

We then clearly have

Theorem 6.2. *Let G be a finitely generated torsion free nilpotent group with Mal'cev completion $G^{\mathbb{Q}}$. If*

$$\zeta_G^{iso}(s) = \sum_{n\in\mathbb{N}} a_n n^{-s} \text{ and } \zeta_G^{up,iso}(s) = \sum_{n\in\mathbb{N}} b_n n^{-s}$$

and if we define

$$\eta_G(s) = \sum_{n\in\mathbb{N}} a_n b_n n^{-s},$$

then

$$\eta_G(s) = \zeta_G^{side,iso}(s)\zeta_G^{up,iso}(s)$$

The proof is exactly the same as that of C_∞^d.

We now wish to calculate the sideways zeta function of $H = F_2^2$. However we are not in the pleasant situation where every overgroup and subgroup is isomorphic to a (unique) known group. However we do know that if $G \leq_f H$ or $H \leq_f G \leq H^{\mathbb{Q}}$

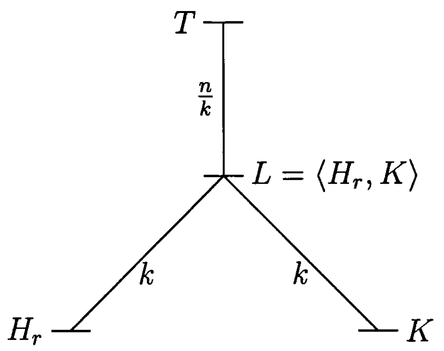


Figure 6.2: The case for H_r

then we will have $G \in \mathcal{H}$, *i.e.* $G \simeq H_t = \langle x, y | [x, y] = z^t, z \text{ central}\rangle$ for some $t \in \mathbb{N}$. Previously we have also calculated the 'partial' zeta functions

$$\phi_{H_r}^{H_t}(s) = \sum_{\substack{G \leq_f H_r \\ G \simeq H_t}} |H_r : G|^{-s} \text{ and } \phi_{H_r}^{\mathrm{up}, H_t}(s) = \sum_{\substack{H \leq_f T \leq H_r^{\mathbb{Q}} \\ T \simeq H_t}} |T : H_r|^{-s}$$

for given $r, t \in \mathbb{N}$. We also observe that

$$\zeta_{H_r}(s) = \sum_{t \in \mathbb{N}} \phi_{H_r}^{H_t}(s) \text{ and } \zeta_{H_r}^{\mathrm{up}}(s) = \sum_{t \in \mathbb{N}} \phi_{H_r}^{H_t, \mathrm{up}}(s).$$

If we now return to trying to use the above counting method we are in the situation as in Figure 6.2 and need to know

- the number of overgroups T of H_r of co-index n (*i.e.* knowledge of $\zeta_{H_r}^{\mathrm{up}}(s)$)
- the number of subgroups K of T of index n (*i.e.* knowledge of $\zeta_T(s)$)
- the number of overgroups M of L of co-index n/k (*i.e.* knowledge of $\zeta_L^{\mathrm{up}}(s)$)

Hence it would be good to fix an isomorphism class for L and another for T and then sum over all possible cases (noting that H_r is fixed and we need no information K in this scenario). However, for the argument above to make sense we must have both T and L in the same isomorphism class, and this clearly would not allow us to obtain $\zeta_{H_r}^{side}(s)$. The next best situation is to fix an isomorphism class for L and allow the isomorphism class of T to be free. However, again, this also would not allow us to use the recursive formula as we would not count all groups K as some of the would not generate a group isomorphic to H_r. Thus we must let L be free.

Now let,

$$\begin{aligned}\zeta_{H_r}(s) &= \sum_{n\in\mathbb{N}} a_n^t n^{-s} \\ \phi_{H_r}^{H_t,up} &= \sum_{n\in\mathbb{N}} B_n^t n^{-2}\end{aligned}$$

and then we make the following definition,

Definition. *Let $H_r, H_t \in \mathcal{H}$, that is $H_r = \langle x, y, |[x,y] = z^r, z$ central $\rangle$ and $H_t = \langle x, y, |[x,y] = z^t, z$ central $\rangle$. The* partial sideways zeta function *of H_r is defined*

$$\phi_{H_r}^{H_t}(s) = \sum_{\substack{K \leq H_r^{\mathbb{Q}} \\ |H_r : K|_{side} < \infty \\ \langle H_r, K\rangle \simeq H_t}} |H_r : K|_{side}^{-s}.$$

If we assume that

$$\phi_{H_r}^{H_t}(s) = \sum_{n\in\mathbb{N}} c_n^t n^{-s}$$

we can begin to find c_n^t. Firstly the number of groups K as in Figure 6.2 is

$$\sum_{t\in\mathbb{N}} b_n^t a_n^t.$$

(that is; we look at all possible overgroups of H_r of co-index n — there are $\sum_{t\in\mathbb{N}} b_n^t$ of these — and consider for each of these groups the number of subgroups of index n; this depends on the isomorphism class t of the overgroup and is a_n^t.)

Next we wish to count this number in a different way. If we consider $L = \langle H_r, k\rangle$. The we have $L \simeq H_t$ for some $t \in \mathbb{N}$. We know that this group has $b_{n/k}^t$ overgroups of co-index n/k where $|L : K| = |L : G| = k$. Thus the contribution of such groups to the count above is

$$\sum_{k|n} c_k^t b_{n/k}^t.$$

If we now sum over all t we find that

$$\sum_{t\in\mathbb{N}} b_n^t a_n^t = \sum_{t\in\mathbb{N}} \sum_{k|n} c_k^t b_{n/k}^t.$$

If we now multiply both sides by n^{-s} and sum over all $n \in \mathbb{N}$ we obtain

$$\sum_{n\in\mathbb{N}}\left(\sum_{t\in\mathbb{N}} b_n^t a_n^t n^{-s}\right) = \sum_{n\in\mathbb{N}}\left(\sum_{t\in\mathbb{N}}\sum_{k|n} c_k^t b_{n/k}^t n^{-s}\right).$$

Now,

$$\sum_{n\in\mathbb{N}}\sum_{t\in\mathbb{N}} b_n^t a_n^t n^{-s} = \sum_{t\in\mathbb{N}}\left(\sum_{n\in\mathbb{N}} b_n^t a_n^t n^{-s}\right)$$

and

$$\sum_{n\in\mathbb{N}}\left(\sum_{t\in\mathbb{N}}\sum_{k|n} c_k^t b_{n/k}^t n^{-s}\right) = \sum_{t\in\mathbb{N}}\left(\sum_{n\in\mathbb{N}}\sum_{k|n} c_k^t b_{n/k}^t n^{-s}\right)$$

so that defining

$$\eta_{H_r}^{H_t}(s) = \sum_{t\in\mathbb{N}} b_n^t a_n^t$$

we have

$$\sum_{t\in\mathbb{N}}\left[\eta_{H_r}^{H_t}(s) - \phi_{H_r}^{H_t side}\phi_{H_r}^{H_t up}\right] = 0.$$

6.1 Other sideways distances

We now give two other possible definitions for the sideways zeta function of a group. The first we have already observed above.

Definition. *Let $G \in \mathcal{T}$ with*

$$\zeta_G(s) = \sum_{n\in\mathbb{N}} a_n n^{-s} \text{ and } \zeta_G^{up}(s) = \sum_{n\in\mathbb{N}} b_n n^{-s}$$

then we define

$$\zeta_G^{\prime side}(s) = \sum_{n\in\mathbb{N}} a_n b_n n^{-s}.$$

This is clearly not a good definition as we count some groups twice.

Another more credible definition of a sideways zeta function is as follows. Firstly we have

Definition. *Let $G \in \mathcal{T}$ and $G^{\mathbb{Q}}$ be its Mal'cev completion. We say a group $L \leq G^{\mathbb{Q}}$ has* sideways distance *n from G if $R = G \cap L$ is such that*

1. $|G:R| = n$

2. $|L:K| = n$

We denote this as $|G:L|'_{side}$.

We then make the following definition.

Definition. *Let* G *be a finitely generated torsion free nilpotent group. The* sideways zeta function *of* G *in* $G^{\mathbb{Q}}$ *is*

$$\zeta_G^{\prime\prime side}(s) = \sum_{\substack{K \leq G^{\mathbb{Q}} \\ |G:K|'_{side} < \infty}} |G:K|'^{\,-s}_{side}.$$

It is clear that this is an equivalent notion to the above. In fact, by suitably changing the above arguments (so instead of overgroups we consider subgroups) we can easily prove the following.

Theorem 6.3. *Let* $G = C_\infty^d$ *and let*

$$\zeta_G(s) = \sum_{n \in \mathbb{N}} a_n n^{-s} \text{ and } \zeta_G^{up}(s) = \sum_{n \in \mathbb{N}} b_n n^{-s}.$$

Then we have

$$\eta_G''(s) = \zeta_G^{\prime\prime side}(s) \zeta_G(s)$$

where

$$\eta_G''(s) = \sum_{n \in \mathbb{N}} a_n b_n n^{-s}.$$

6.2 Conclusions

We now attempt to give an overview of this thesis, mentioning some results and possible future directions for research.

In the last twenty of so years there has been a large amount of research into the area of subgroup growth. This began by focusing on torsion free finitely generated nilpotent groups (or $\mathcal{T}$-groups) but has also lately considered other non-nilpotent groups. The subgroup growth of a $\mathcal{T}$-group is calculated using various methods and thus encoded in the zeta function of a group,

$$\zeta_G(s) = \sum_{H \leq_f G} |G : H|^{-s} = \sum_{n \in \mathbb{N}} a_n n^{-s}$$

where $a_n = |\{H \leq_f G \mid |G : H| = n\}|$. We began this thesis by considering the zeta functions, that were already known, for the free abelian group, the free nilpotent group on two generators of class 2 – sometimes called the Heisenberg group – and groups which are commensurable with F_2^2. We also calculated zeta functions for certain abelian groups with some torsion.

A $\mathcal{T}$-group G can be embedded into its Mal'cev completion $G^{\mathbb{Q}}$. With the subgroup case in mind and this 'universe' to work in we turned the problem on its head and considered the idea of supergroup or overgroup growth. That is instead of counting subgroups of finite index and encoding this information using a zeta function, we counted the number of overgroups of finite co-index and defined the co-zeta function of a group

$$\zeta_G^{\mathrm{up}}(s) = \sum_{G \leq_f T \leq G^{\mathbb{Q}}} |T : G|^{-s} = \sum_{n \in \mathbb{N}} b_n n^{-s}$$

where $b_n = |\{G \leq_f T \leq G^{\mathbb{Q}} \mid |T : G| = n\}|$.

Focusing on torsion free finitely generated abelian groups and nilpotent groups (especially F_2^2) we adapted the 'good basis' method for counting subgroups into a method for counting overgroups. We then calculated the co-zeta function for the $\mathcal{T}$-groups for which we already had zeta functions. The results that were obtained were surprising in that they we often 'nicer' than the zeta functions for the subgroup case. An example of this is the zeta and co-index zeta function for F_2^2 which we give below

$$\begin{aligned} \zeta_{F_2^2}(s) &= \frac{\zeta(s)\zeta(s-1)\zeta(2s-2)\zeta(2s-3)}{\zeta(3s-3)} \\ \zeta_{F_2^2}^{\mathrm{up}}(s) &= \zeta(s)\zeta(2s-1)\zeta(2s-2). \end{aligned}$$

The idea of encoding the number of subgroups as coefficients of a zeta function comes from number theory and the Dedekind zeta function of a number field. As such there is no direct group theoretic translation, so as well as counting subgroups, counting normal subgroups or subgroups isomorphic to the original group is often considered. The isomorphism case is interesting, because of the fact that if all subgroups will belong to a countable class of isomorphism types the zeta function can be calculated by enumerating these 'partial' cases (subgroups of a group which belong to a particular isomorphism class) and then adding these results together. In the same vein we calculated the co-normal and co-isomorphism function for the groups we already had co-index functions for, as well as in some more general cases.

It is here that the most surprising result occurs. Let F_n^c be the free nilpotent group of class c on n generators, then Theorem 4.20 tells us that

$$\zeta_{F_n^c}^{\triangleleft,\mathrm{up}}(s) = \zeta_{\mathbb{Z}^d}(s)$$

where $d = \mathrm{rank}(Z(F_n^c))$. This in itself this seems unsurprising. However the proof of the theorem reveals that this gives us a converse to Lemma 4.16 (which states that an overgroup T of G generated by G and a finite co-index overgroup of the centre of G has G as a normal subgroup) when we are dealing with free nilpotent groups. This is a truly surprising result and one which deserves a much more elegant proof. This highlights one of the more negative aspects of this thesis, in that some of the proofs are very technical and do not give an indication why certain things, like the above result are happening. This is one direction which could be considered in the future.

We also attempted to calculate the co-index zeta functions of some non-nilpotent groups, but unsuccessfully. Looking at the crystallographic groups we attempted to define a universe in which to work (as all these groups bar one are not nilpotent and so do not have a Mal'cev completion). This universe was not perfect in that the universe we used ignored growth in one 'direction.' However, the attempt did not succeed (in that there were countably may overgroups of some indices) possibly because of the universe which was chosen to work within. The definition of such a 'good' universe for such groups is possibly one worth considering in the future.

Another direction to be considered in the future is which groups have polynomial growth. This result is known in the case of subgroups,

Theorem. *[Lub93, Theorem 4.1] Let G be a finitely generated residually finite group. Then G has polynomial subgroup growth if and only if G is virtually solvable of finite rank.*

A similar result, at least for the case of $\mathcal{T}$-groups would be something to consider for overgroup growth.

Another result which would be worth pursuing is whether the co-index zeta function of a $\mathcal{T}$-group is rational. Again, this result is known in the zeta function case with a result of Grunewald, Segal and Smith [GSS88],

Theorem. *[GSS88] or see [Lub93, Theorem 2.19] The p-th Euler factor of the zeta function of a finitely generated torsion -free nilpotent group is a rational function of* p^{-s}.

(In fact there is a more general result than this – however as we only have co-index zeta functions for $\mathcal{T}$-groups we omit this.)

Finally, in this thesis we considered the notion of 'sideways' growth and sideways zeta functions. The results here are not as 'clean' as for those in the co-index or index case. However, the tools were available and so some calculations were performed. Growth issues for this zeta function are clearly going to be worse, in that a group to the side is a result of an overgroup and then a subgroup. This is a possible area for the future, but more interesting avenues seem to exist.

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Appendix A

Other Work

Before embarking on the work which forms the bulk of this thesis, I investigated articles by D. L. Johnson and G. C. Smith on the theory of monoid presentations. Here I present the results of this work which I did not pursue beyond this stage. This work is independent of the earlier material in this thesis is developed from a final year undergraduate project and so the explanation may be a little naive in places.

ATW

A.1 Basic Concepts

A.1.1 Set-Theoretic Preliminaries

We shall start by defining some basic concepts. We will assume that the reader has a basic understanding of sets and set membership as well as operations of union, intersection and difference of sets. The empty set will be denoted $\emptyset$. If A is a subset of B we will write $A \subseteq B$ whether or not the inclusion is proper. The *Cartesian product* of sets X and Y is the set $X \times Y$ of all ordered pairs (x, y) with $x \in X$ and $y \in Y$.

A *relation* $\sim$ from X to Y is a subset $\mathfrak{R}$ of $X \times Y$. If $X = Y$ the relation is said to be a relation on X. If (x, y) is in $\mathfrak{R}$ we will write $x \sim y$. An *equivalence relation* on a set X is a relation $\sim$ such that the following hold:

(i) $x \sim x$ for all $x \in X$,

(ii) if $x \sim y$ then $y \sim x$,

(iii) if $x \sim y$ and $y \sim z$ then $x \sim z$.

These conditions are called *reflexive*, *symmetric*, and *transitive* respectively. The equivalence class of x, denoted $[x]$, is the set of all elements equivalent to x. Two equivalence classes are either equal or disjoint. So $\{[x] \mid x \in X\}$ is a partition of X.

Lemma A.1.1. *The intersection of any non-empty collection of equivalence relations is an equivalence relation.*

Proof. Let $\sim$ denote the intersection relation. Thus for $x \in X$, $x \sim x$ as each relation is an equivalence. If $x \sim y$ then in each equivalence relation, $\equiv$, in the intersection $x \equiv y$ and so $y \equiv x$. Thus $y \sim x$. Similarly if $x \sim y$ and $y \sim z$ then in each equivalence relation, $\equiv$, making up the intersection $x \equiv y$ and $y \equiv z$. Hence $x \equiv z$, so $x \sim z$ and $\sim$ is an equivalence. $\square$

A *function* (or *map*) from X to Y is a relation such that for each $x \in X$ there is a unique $y \in Y$ to which it relates. We will write $f : X \to Y$ and refer to X as the domain. If y is the unique element related to x we will write $y = (x)f$. The *natural* (or canonical) map from a set X to its set of equivalence classes is the map that takes x to $[x]$.

We say f is *surjective* if $(X)f = Y$ and that f is *injective* if for all $x_1, x_2 \in X$ then $(x_1)f = (x_2)f$ implies that $x_1 = x_2$. If f is both injective and surjective then it is *bijective.*

A.1.2 Groups, Monoids and Presentations

Let S be a set and $\bullet$ a closed associative binary operation on S. A *monoid* is a pair $(S, \bullet)$ where there exists an element $e \in S$ such that for all $x \in S, x \bullet e = e \bullet x = x$. Here e is called the *identity element* of S. A *unit* $u \in S$ is an element such that there exists a $v \in S$ with the property that $u \bullet v = v \bullet u = e$. Here v is called the *inverse* of u and denoted u^{-1}. A *group* is a pair $(S, \bullet)$ such that every element of S is a unit. Note that henceforth we will use multiplicative notation for binary operations, unless this will make things confusing. So $x \bullet y$ will become xy and the identity element, e, will be denoted 1. We will refer to S as a group or a monoid without reference to the operation.

Let M be a monoid with identity element 1. A *submonoid* of M is a subset N of M such that the following hold:

(i) $1 \in N$,

(ii) if $x \in N$ and $y \in N$ then $xy \in N$.

Additionally if N consists of units and their inverses N is a *subgroup* of M. A submonoid is a monoid under restriction to N of the binary operation and similarly for a subgroup.

Lemma A.1.2. *Let M be a monoid. The intersection of any non-empty collection of submonoids of M is a submonoid. The intersection of any non-empty collection of subgroups of M is a subgroup.*

Proof. Let N denote the intersection. Clearly 1 is in N. If $x \in N$ and $y \in N$ then x and y belong to all the submonoids in the intersection. So their product xy belongs to all the submonoids and is in N. So N is a submonoid of M. A similar argument applies for subgroups. $\square$

Let Y be a subset of M and contemplate $N = \bigcap_{i \in I} N_i$, where the N_i are all submonoids of M which contain Y and I is an indexing set. Observing that as $Y \subseteq M, I \neq \emptyset$ so by the preceding lemma N is a submonoid of M. Clearly N is the smallest submonoid of M which contains Y. We call this the submonoid generated by Y and write $N = \text{Mon}< Y >$.

A similar argument can be applied to a subset of units (and their inverses) and subgroups containing Y and this would result in the subgroup generated by Y, denoted $H = \text{Grp}< Y >$.

Let us now consider an example of a monoid. Let X be a set. A *word* is a finite sequence $U = u_1, u_2, \ldots, u_n$ with each $u_i \in X$. The empty sequence (where $n = 0$) is denoted ε. Writing $u_1 u_2 \ldots u_n$ for U and identifying the elements of X with the corresponding words of length one we can form a monoid using the (associative) operation of juxtaposition. We call this set of words X^*. If A, B

and C are non-empty words in X^* with $U = ABC$ then A is a *prefix* of U, C a *suffix* of U and B a *subword* of U. A *syllable* of a word is a non-empty subword of maximal extent subject to consisting of only a single letter.

Let M and N be monoids with identities 1_M and 1_N respectively. A *homomorphism* from M to N is a function $f : M \to N$ such that the following conditions hold:

(i) $(xy)f = (x)f(y)f$

(ii) $(1_M)f = 1_N$

If M and N are groups then (ii) follows from (i) as :

$$(x)f \cdot 1_N = (x)f = (x \cdot 1_M)f = (x)f \cdot (1_M)f$$

and so pre-multiplication by $((x)f)^{-1}$ will yield $1_N = (1_M)f$. It can also be easily shown that $((x)f)^{-1} = (x^{-1})f$ and so if M is a group then $(M)f$ is as well.

An *epimorphism* is a surjective homomorphism. An *isomorphism* is an bijective homomorphism.

A monoid F is said to be *free on a subset* $X \subseteq F$ if given any monoid M and a map $\theta : X \to M$ there is a unique homomorphism $\theta' : F \to M$ extending θ, that is having the property that $(x)\theta' = (x)\theta$ for all $x \in X$. For any set X the monoid X^* is called the *free monoid* generated by X. We justify this definition by the following:

Lemma A.1.3. *Let X be a set and let M be a monoid. For each function $f : X \to M$ there is a unique extension F of f to a homomorphism of X^* into M.*

Proof. Let $U = u_1 u_2 \dots u_n$ be in X^*, with each $u_i \in X$. The only way to define $(U)F$ so that F becomes a homomorphism from X^* to M is to set $(U)F = (u_1)f \dots (u_n)f$. Thus $(UV)F = (U)F(V)F$ by induction and the identity, as it is the product of no elements, maps to 1. So F is a homomorphism. $\square$

Let M be a monoid. A *congruence* on M is an equivalence relation $\sim$ on M which is compatible with the multiplication in M, *i.e.* if $x, y, z \in M$ with $x \sim y$ then $xz \sim yz$ and $zx \sim zy$.

Lemma A.1.4. *Let M be a monoid and $S \subseteq M \times M$. The intersection $\sim$ of all congruence relations on Mcontaining S is a congruence. $\sim$ is called the congruence generated by S.*

Proof. There is at least one congruence containing S, namely $M \times M$. By Lemma A.1.1, $\sim$ is an equivalence relation. Suppose $x \sim y$, then for each $z \in M$ and congruence, $\equiv$, containing S we know that $x \equiv y$ so $xz \equiv yz$ and $zx \equiv zy$. Thus $xz \sim yz$ and $zx \sim zy$ □

Let Q be the set of equivalence classes of $\sim$ and let $[x]$ be the equivalence class of x for all $x \in M$. The rule $[x][y] = [xy]$ defines an associative binary operation on Q with $[1]$ as the identity. Thus Q is a monoid, the *quotient monoid* of M modulo $\sim$.

Let X be a set and let $\mathfrak{R}$ be a subset of $X^* \times X^*$. The monoid $Q = \text{Mon}< X \mid \mathfrak{R} >$ is defined to be the quotient monoid of X^* modulo the congruence relation generated by $\mathfrak{R}$. The pair $(X, \mathfrak{R})$ is said to be a *monoid presentation* for Q and any monoid isomorphic to Q. The presentation is finite if both X and $\mathfrak{R}$ are finite. A monoid M is finitely presented if M has finite presentation. If $X = \{x_1, \ldots, x_n\}$ and $\mathfrak{R}$ consists of pairs (U_i, V_i) for $1 \leq i \leq t$ then Q is sometimes written

$$Q = \text{Mon}< x_1, \ldots, x_n \mid U_1 = V_1, \ldots, U_t = V_t >.$$

Here the equations $U_i = V_i$ are the *defining relations* for Q. Under the natural map from X^* to Q the words U_i and V_i will map to the same element. Elements of $\text{Mon}< X \mid \mathfrak{R} >$ are equivalence classes of words. The equivalence class of U is usually denoted $[U]$

A.2 Knuth-Bendix Procedure for Strings

We now aim to describe the Knuth-Bendix procedure for strings [KB70]. This procedure manipulates rewriting systems, which are pairs of words. Later this will help us to decide on whether a given monoid presentation is a group or not.

Let $M = \text{Mon}< X^* \mid \mathfrak{R} >$. The Knuth-Bendix procedure takes a translation invariant well ordering of X^* and tries to produce a complete rewriting system for M. This procedure is not guaranteed to terminate.

A.2.1 Orderings of free Monoids

Let S be a set. A *linear ordering* of S is a transitive relation $\prec$ on S such that for any two elements s and t of S one of the following holds: $s \prec t, s = t$ or $s \succ t$. The ordering $\prec$ is a *well ordering* if there are no infinite decreasing sequences (s_n) in S (that is $s_i \succ s_{i+1}$ for all $i \geq 1$). An ordering $\prec$ is a *translation invariant* ordering if $U \prec V$ implies that $AUB \prec AVB$ for all $A, B \in X^*$.

It is easy to define a linear ordering of a finite set: we just list the elements in some (arbitrary) order. As there can be no infinite decreasing sequences this will always be a well ordering. A set of size n will have $n!$ possible orderings. However for infinite sets things are not so easy, as not all linear orderings will be well-orderings.

We now define the ordering which will interest us most. The *short-lex* or more correctly the *length-plus-lexicographic* ordering. Here lexicographic means left-to-right-lexicographic. The lexicographic ordering is as follows: given an ordering $\prec$ on X and for $s = s_1 s_2 \dots s_m, t = t_1 t_2 \dots t_n \in X^*$, with each s_i and t_i in X, we say $s \prec t$ if and only if one of the following holds:

(a) $m < n$ and $s_i = t_i$, $1 \leq i \leq m$,

(b) there is an i with $1 \leq i \leq \min(m, n)$ such that $s_j = t_j$ for $1 \leq j < i$ and $s_i \prec t_i$.

This is not a well-ordering because if $a, b \in X$ with $a \prec b$ then $ab \succ a^2b \succ a^3b \succ \cdots$ is a strictly decreasing infinite sequence in X^*. The short-lex ordering is an improvement on this, with $u_1 \dots u_m \prec v_1 \dots v_n$ provided either that $m < n$ or that $m = n$ and $u_1 \dots u_m$ comes before $v_1 \dots v_n$ lexicographically.

The following justifies our use of short-lex as a translation invariant well ordering.

Lemma A.2.6. *If $\prec$ is a linear ordering of S then the corresponding lexicographic ordering is a linear ordering of S^n.*

Proof. Let $s = (s_1, s_2, \dots, s_n) \in S^n$ and $t = (t_1, t_2, \dots, t_n) \in S^n$. Then we have either $s_1 \prec t_1, s_1 = t_1$ or $s_1 \succ t_1$ as $\prec$ is a linear ordering of S. Now if $s_1 \neq t_1$ we are done as $s \prec t$ or $t \prec s$ depending on whether $s_1 \prec t_1$ or $s_1 \succ t_1$. Assume now that $s_j = t_j$ for all j, $1 \leq j \leq k < n$. Now as $\prec$ is a linear ordering on S, we have $s_{k+1} \prec t_{k+1}, s_{k+1} = t_{k+1}$ or $t_{k+1} \prec s_{k+1}$. Thus if $s_{k+1} \neq t_{k+1}$ we are done by induction. If $k = n-1$ and $s_{k+1} = t_{k+1}$ we have finished and $s = t$. $\square$

Lemma A.2.7. *If $\prec$ is a linear ordering on X then the lexicographic ordering is a linear ordering on X^*.*

Proof. Let $U = u_1 \dots u_m$ and $V = v_1 \dots v_n$ be words in X^*. If $|U| = |V|$ apply Lemma A.2.6 with $n = |U|$ and we are done. Now assume (without loss of generality) that $|U| < |V|$. Then if $V = UA$ for some non-empty A, we have $U = V$. If not apply Lemma A.2.6 to $(u_1, \dots, u_{|U|})$ and $(v_1, \dots, v_{|U|})$. If $(u_1, \dots, u_{|U|}) \prec (v_1, \dots, v_{|U|})$ we have $U \prec V$ in X^* otherwise $V \prec U$ (note $U \neq V$ as we have already dealt with this case). Thus $\prec$ is a linear ordering of X^*. $\square$

Lemma A.2.10. *The short-lex ordering is a linear ordering of X^*. If $\prec$ is a well ordering of X then X^* is also well ordered.*

Proof. Linearity is a consequence of the previous lemma. Hence we only have to show well ordering. Let $U_1 \succ U_2 \succ U_3 \succ \cdots$ be strictly decreasing sequence of words in X^*. Since $|U_i| \geq |U_{i+1}|$ for all i, from some point on all the U_i will have the same length m. By Lemma A.2.6, X^m is well ordered and hence the sequence must terminate. So $\prec$ is a well ordering of X^*. $\square$

Proposition A.1. *The short-lex ordering of X^* is translation invariant.*

Proof. Let $U = u_1 \dots u_n$ and $V = v_1 \dots v_m$ and suppose that $U \prec V$. To prove translation invariance it suffices to prove that $Ux \prec Vx$ and $xU \prec xV$ for all x in X. If $n < m$ then $|Ux| = |xU| < |Vx| = |xV|$. Thus we may assume that

$m = n$. Let U and V differ first in the i-th term. Then Ux and Vx also differ first in the ith term (as Ux and Vx have the same first terms). Thus $Ux \prec Vx$. A similar argument gives $xU \prec xV$. $\square$

We will call a translation invariant well ordering a *reduction ordering.*

A.2.2 Canonical Forms and Rewriting systems

Suppose we are working with a finitely presented monoid M. Although it is generally impossible to tell when two given words are in the same equivalence class (the *word problem*) it is natural to be optimistic and look for a solution in M anyway. Assume we know that M is a quotient monoid of the free monoid. One approach to the word problem is to choose for each u in M one word U which defines u. Such a choice is a *canonical form* for u.

Usually we want the canonical form for U to be the "simplest" word defining U. As we saw previously a reduction ordering provides a convenient notion of simplicity. Thus fixing a reduction ordering $\prec$ on M, consider the set of words that define u. This is non-empty and so will have a smallest element U. We define this U to be the *canonical form* for u relative to $\prec$.

Let (P, Q) be an element of the generating set $\mathfrak{R}$ for $\sim$. As replacing (P, Q) by (Q, P) does not change $\sim$, we may assume that $P \succ Q$. In this case (P, Q) is called a rewriting rule with respect to $\prec$. If every element of $\mathfrak{R}$ is a rewriting rule then $\mathfrak{R}$ is called a rewriting system with respect to $\prec$.

Thus we can rewrite words in M, by the following process. Let U be a word in M. If (P, Q) is in $\mathfrak{R}$ then if $U = APB$ we can write $V = AQP$ and we have $U \sim V$ and $V \prec U$. Since $\prec$ is a well ordering this process cannot go on indefinitely and will eventually produce a word C in M which contains no P's such that P is the left side of a rewriting rule. This C is called *irreducible* or *reduced* with respect to $\mathfrak{R}$. To emphasise that rewriting replaces P by Q we will often write $P \to Q$ or $P \to_{\mathfrak{R}} Q$.

Let us consider two examples. Let $X = \{a, b, c, d\}$ and let $\mathfrak{R}$ consist of the rules $ab \to 1$, $ba \to 1$, $cd \to 1$ and $dc \to 1$. Let $M = \mathrm{Mon}< X \mid \mathfrak{R} >$ and $U = acbadbdcac \in M$. We can rewrite U in several ways using $\mathfrak{R}$ and here are two:

$$ac\underline{ba}dbdcac \to a\underline{cd}bdcac \to \underline{ab}dcac \to \underline{dc}ac \to ac$$

also:

$$acbadb\underline{dc}ac \to acbad\underline{ba}c \to acba\underline{dc} \to ac\underline{ba} \to ac$$

Now let $X = \{a, b\}$ and $\mathfrak{R}$ consist of the rules $a^2 \to 1$, $b^{10} \to 1$ and $ba \to ab^4$. Let us rewrite baa in two ways. The first is easy:

$$b\underline{aa} \to b$$

The second takes a little longer:

$$\begin{gathered}\underline{ba}a \to abbb\underline{ba} \to abb\underline{ba}bbbb \to ab\underline{ba}bbbbbbbb \to \\ ababb\underline{bbbbbbbbbb} \to a\underline{ba}bb \to \underline{aa}bbbbbb \to bbbbbb\end{gathered}$$

The result is b in the first case and b^6 in the second. We will not delve deeply into the subject of rewriting systems. Our only goal will be to develop a condition on a rewriting system $\mathfrak{R}$ which guarantees that the result V given by a rewriting process depends only on the input word U and not on the choices we make during rewriting.

Let $\mathfrak{R}$ be a rewriting system on X^* with respect to a reduction ordering $\prec$ and let $\sim$ be the congruence generated by $\mathfrak{R}$. We say that $V \in X^*$ is *derivable from* $U \in X^*$ *using* $\mathfrak{R}$ if there is a sequence

$$U = U_0, U_1, \ldots, U_t = V$$

with $t \geq 0$ such that U_{i+1} is derivable from U_i in one step for $0 \leq i \leq t$. In this case we shall write $U \to_{\mathfrak{R}}^* V$. Clearly $\to_{\mathfrak{R}}^*$ is reflexive and transitive. Also if $U \to_{\mathfrak{R}}^* V$ then $U \sim V$ and $U \succeq V$.

Lemma A.2.12. *If $U \to_{\mathfrak{R}}^* V$ then $UW \to_{\mathfrak{R}}^* VW$ and $WU \to_{\mathfrak{R}}^* WV$ for all $W \in X^*$.*

Proof. If $U = U_0, U_1, \ldots, U_t = V$ is such a sequence then $WU = WU_0, WU_1, \ldots, WU_t = WV$ will be also. Similarly for UW. □

Lemma A.2.13. *If $U, V \in X^*$ then $U \sim V$ if and only if there is a sequence $U = U_0, U_1, \ldots, U_t = V$ such that for $0 \leq i \leq t, U_i \to_{\mathfrak{R}}^* U_{i+1}$ or $U_{i+1} \to_{\mathfrak{R}}^* U_i$.*

Proof. Let use write $U \equiv V$ is such a sequence exists. Since $U_i \sim U_{i+1}$, we know that $U \equiv V$ implies that $U \sim V$. It is easy to see that $\equiv$ is an equivalence relation. By the preceeding lemma $\equiv$ is a congruence. If (P, Q) is in $\mathfrak{R}$, then $P \equiv Q$. Therefore $\sim$ is contained in $\equiv$. So $\sim$ and $\equiv$ are the same relation. □

There are a number of useful properties that $\rightarrow^*_{\mathfrak{R}}$ can have:

- **Church-Rosser property** if $U \sim V$ then there is a word Q such that $U \rightarrow^*_{\mathfrak{R}} Q$ and $V \rightarrow^*_{\mathfrak{R}} Q$.
- **Confluence** if $W \rightarrow^*_{\mathfrak{R}} U$ and $W \rightarrow^*_{\mathfrak{R}} V$ then there is a word Q such that $U \rightarrow^*_{\mathfrak{R}} Q$ and $V \rightarrow^*_{\mathfrak{R}} Q$.
- **Local confluence** if $W \rightarrow_{\mathfrak{R}} U$ and $W \rightarrow_{\mathfrak{R}} V$ then there is a word Q such that $U \rightarrow^*_{\mathfrak{R}} Q$ and $V \rightarrow^*_{\mathfrak{R}} Q$.

We now have :

Proposition A.2. *If the Church-Rosser property holds then every $\sim$-class contains a unique irreducible element.*

Proof. Suppose U and V are reduced and in the same $\sim$-class. Then by the Church-Rosser property there exists a Q such that $U \rightarrow^*_{\mathfrak{R}} Q$ and $V \rightarrow^*_{\mathfrak{R}} Q$. However since U and V are reduced we must have $U = Q = V$. □

We state the following:

Proposition A.3. *For any rewriting system $\mathfrak{R}$ relative to a reduction ordering $\prec$ the Church-Rosser property, confluence and local confluence are equivalent.*

Finally we have:

Proposition A.4. *If $\mathfrak{R}$ is a confluent rewriting system on X^* then after rewriting $U \in X^*$ the value of the reduced word depends only on $\mathfrak{R}$ and U and not the choices made during the rewriting.*

Proof. By the previous two propositions the value after rewriting is the unique canonical form of the $\sim$-class containing U. □

A.2.3 A Test for confluence

Let $\prec$ be a reduction ordering on X^* and let $\mathfrak{R}$ be a finite rewriting system on X^* with respect to $\prec$. If $\mathfrak{R}$ is confluent we can solve the word problem in $M = \text{Mon}< X \mid \mathfrak{R} >$ by computing canonical forms. In this section we will describe a test for confluence.

We will use the following theorem:

Theorem A.5. *Suppose local confluence fails at a word W but does not fail at any proper subword of W. Then one of the following conditions holds:*

1. *W is the left side of two different rules in $\mathfrak{R}$,*
2. *W is the left side of a rule in $\mathfrak{R}$ and W contains another left side as a proper subword,*
3. *W can be written ABC, where A, B and C are non-empty words and AB and BC are left sides in $\mathfrak{R}$.*

If W satisfies the conclusion of the previous theorem then we shall say that W is an overlap of left sides in $\mathfrak{R}$. If $\mathfrak{R}$ is finite then the set of words which are overlaps of left sides in $\mathfrak{R}$ is finite. For each of these words W we can list the finite set of words $\mathfrak{U}$ derivable from W in one step. If we then rewrite each word in $\mathfrak{U}$ until it is in its irreducible form and obtain more than one value, $\mathfrak{R}$ is not confluent for we have found two words irreducible with respect to $\mathfrak{R}$ that define the same value in M. If we obtain the same value for all words in $\mathfrak{U}$ local confluence does not fail at W. By performing this test for all words derivable from W in one step we can decide whether or not $\mathfrak{R}$ is confluent.

A.2.4 The Knuth-Bendix Procedure

The Knuth-Bendix procedure tries to find a confluent and finite rewriting system (we shall call this a *complete* rewriting system).

Let $(X, \mathfrak{R})$ be a finite monoid presentation, let $\prec$ be a reduction ordering on X^*, and let $\mathfrak{T}$ denote the complete rewriting system.

A brief description of the algorithm is as follows. Firstly redirect all the rules in $\mathfrak{R}$ via $\prec$. Now list all the overlaps of the left hand sides of the rewrite rules, marking any overlap "done" that we have already resolved. Select any unmarked pair with overlap not previously studied and consider this overlap: we look at the two ways of rewriting this word and then simplify the resulting words into U and V say. Adding, if necessary, the new rule $U \to V$ if $U \succ V$ or $V \to U$ if $U \prec V$. Mark this overlap as done, and update the lists. Repeat this until all overlaps are marked as done. Note that it is possible to generate an infinite system as each new rule may generate new overlaps.

We now present a basic version of the procedure written in more suitable language. Note however that this version is not practical for serious computation and may be improved, see [Sim94] for more details.

Two subroutines are used in the Knuth-Bendix procedure. The subroutine TEST adds a new rule, if necessary, in order to ensure that there is a word derivable from two given words using the rules in $\mathfrak{S} = \{(P_i, q_i) \mid 1 \leq i \leq n\}$ (the set of current rules).

```
TEST(U,V)                       ; U,V words in X*
    REWRITE(U,A)                ; REWRITE applies rules to U
                                ; and returns the result in A
    REWRITE(V,B)

    if A ≠ B
        if A ≺ B swap A and B
        n = n + 1, P_n = A, q_n = B
```

The second subroutine, OVERLAP, is a test for confluence. It checks the overlaps of P_i and P_j in which P_i occurs at the beginning of the word.

```
OVERLAP(i,j)                    ; i, j integers not exceeding n.
    For k = 1 to |P_i|
        Let P_i = AB with |B| = k
        Let U be the longest common prefix of B and P_j
        let B = UD and P_j = UE
```

if D or E is empty then $\textsc{Test}(AQ_jD, Q_iE)$

Then here is the Knuth-Bendix procedure for strings:

```
KBS(X ≺, 𝔑, 𝔗)                 ; X is the finite set,
                               ; ≺ a reduction ordering on X*,
                               ; 𝔑 a finite subset of X* × X*
                               ; 𝔗 complete rewriting system, if it is finite
; WARNING TERMINATION MAY NOT OCCUR
    n = 0, i = 1
    For (U, V) in 𝔑
        TEST(U,V)

    While i ≤ n
        For j = 1 to i
            OVERLAP(i,j)
            if j < i
                OVERLAP(j,i)
        i = i + 1

    Let 𝔓 be the set of P_i such that every proper
        subword of P_i is irreducible with respect to 𝔖
    𝔗 = ∅

    For P in 𝔓
        REWRITE(P,Q)
        Add (P, Q) to 𝔗
```

Finally we state Knuth-Bendix theorem,

Theorem A.6 (Knuth-Bendix). *If the complete reduced rewriting system is finite the* KBS *$(X, \prec, \mathfrak{R}, \mathfrak{T})$ will terminate with $\mathfrak{T}$ as the complete reduced rewriting system.*

A.3 Theory of Single Relator Monoids

Now we turn aside from the basics and consider the question of whether a given monoid presentation defines a group or not. We will concentrate on a two letter monoid with a single relator and attempt to summarise two papers recently written on this subject. These papers are by D. L. Johnson [Joh97] and G. C. Smith [Smi97].

Throughout this chapter let F = Mon$< x, y \mid _ >$, *the free monoid of rank 2* and M =Mon$< x, y \mid r = 1 >$ be the *quotient monoid*, the monoid F modulo the equivalence relation generated by $r = 1$.

Firstly we consider the positive cases, that is when we know that a monoid is a group. Some examples of this are when $r = xyyx$, $r = xy^2xy$ (J. M. Howie) and more generally (D. E. Cohen) $r = (xy^l)^m y(xy^l)^n$ where l, m, n are non-negative integers such that $1 \leq m \leq n$. To prove all these cases we appeal to Johnson's first result:

Proposition A.7 (Johnson). *Let $u, v \in F$ begin with x, y respectively and let m, n be integers with $1 \leq m \leq n$. Then M = Mon$< x, y \mid u^m v u^n = 1 >$ is a group.*

Smith observes that the monoid M with an odd number of syllables (strictly greater than one) in the relator must be a group (since it will begin and end with the same syllable). This also covers the case where the relator is a palindrome.

Now we consider the case of when M = Mon$< x, y \mid x^a y^b x^c y^d = 1 >$ Johnson uses the above to deal with some positive cases and then goes onto describe when it isn't a group (see Proposition A.12). Smith on the other hand decides exactly when the above is a group via:

Theorem A.8 (Smith). *Let M be as above with $r = x^a y^b x^c y^d$ with a, b, c and d positive. The monoid M is a group if and only if $d \leq b, a \leq c$ and at least one of these inequalities is strict.*

Smith then progresses further, using the same method of proof to get:

Theorem A.9 (Smith). *Let M be as above with $r = x^a y^b x^c y^d x^e y^f$ with all the exponents positive. The monoid M is a group if and only if*

(a) $a \leq c, b = d, c = e, f \leq b$ *and at least one of the inequalities is strict,*

(b) $a \leq e, f \leq b$ *and at least one of the inequalities is strict,*

(c) $a = e, b = f, c \leq a, b \leq d$ *and at least one of the inequalities is strict,*

(d) $a = e, b = f, a \leq c, d \leq b$ *and at least one of the inequalities is strict.*

Later we shall extend this further. Smith then goes onto a more general result, refuting a conjecture of R. M. Thomas:

Conjecture A.10 (R. M. Thomas). *If* $r \in Mon< x, y, z \mid \_ >$ *and* r *is not a palindrome then* $Mon< x, y, z \mid r = 1 >$ *is not a group.*

Theorem A.11 (Smith). *Let* $n \geq 2$ *be a natural number. There is a single relator monoid* M_n *on* n *letters with the relator not a palindrome but* M_n *a group.*

We now turn our attention to the negative case, when a monoid is not a group. Again we have examples of this. Letting $r = xy$ does not form a group (see [Sim94] p26), and more generally if $r = x^a y^b$ for positive a, b then M is not a group. Cases of relators of zero and one syllable length are obviously not groups.

We demonstrate that $M = \text{Mon}< x, y \mid x^a y^b = 1 >$ is not a group by following the method that Johnson takes. Using the fact that if a homomorphic image of a monoid is not a group then the monoid is not a group, we define a homomorphism from M to Z, the set of all maps from $\mathbb{Z}$ to itself (this is a monoid under composition of maps). Defining $\xi, \eta \in Z$ as follows:

$$(n)\xi = \begin{cases} n + b, & n \geq 0, \\ n, & n < 0, \end{cases} \quad (n)\eta = \begin{cases} n - a, & n \geq a, \\ n, & n < a. \end{cases}$$

It is easy to check that $\xi^a \eta^b = 1$, so that the map

$$\left. \begin{array}{l} M \to Z \\ x \mapsto \xi \\ y \mapsto \eta \end{array} \right\}$$

is a homomorphism. Since ξ is not surjective it has no left inverse in Z. Hence Z is not a group so neither is M.

Looking again at $M = \text{Mon}< x, y \mid x^a y^b x^c y^d >$, Johnson observes that if this is a group then so is $Q = \text{Mon}< x, y \mid x^a y^b = x^c y^d = 1 >$, its homomorphic image.

Thus if we know when Q is not a group we can deduce some information about M.

Proposition A.12 (Johnson). *Q is a group if and only if det* $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \neq 0$.

Looking now at some examples we see that M is not a group when $r = x^2y^2xy$ (via Theorem A.9) and when $r = x^2y^3xy$. Now in the first case we see that

$$\det \begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix} = 0$$

and so Q is not a group. However in the second case

$$\det \begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix} = -1$$

but M is not a group.

Johnson now generalises this proposition:

Proposition A.13 (Johnson). *Let $(k_i, l_i) \in \mathbb{N}^2$, for $i = 1, 2, \ldots, m$, with*

$$\sum_{i=1}^{m} k_i = k \text{ and } \sum_{i=1}^{m} l_i = l.$$

Assuming that

$$(k_1 + k_2 + \cdots + k_i)l \geq (l_1 + l_2 + \cdots + l_i)k$$

for $i = 1, \ldots, m$, the monoid $M = Mon< x, y \mid r = 1 >$ with $r = x^{k_1}y^{l_1} \ldots x^{k_m}y^{l_m}$ is not a group.

We also have two negative results due to Smith, one of which subsumes the other:

Theorem A.14 (Smith). *Let $M = Mon< x, y \mid r = 1 >$ where r involves an even number of syllables. If the strictly highest power of x or y occurring in this product is an end syllable then M is a not group.*

Theorem A.15 (Smith). *Let X be a set of size at least two. If we let $M = Mon< X \mid r = 1 >$ and no proper initial segment of r coincides with a proper terminal segment of r then M is not a group.*

Proofs

We will prove only some of the above theorems and propositions. We start with :

Proof of Proposition A.7. Observing that we need only seek a right inverse (by undergraduate exercise) to y we have:

$$1 = u^m v u^n = u^m v u^{n-m} u^m = u^m v u^{n-m} \cdot u^m v u^n u^m = u^m v u^n v u^n u^m = v u^n u^m$$

Thus as v begins with a y, y will have a right inverse. □

We will prove Theorem A.8 and not Theorem A.9 or A.15 as these are proved using the same method, of which we will see more later.

Proof of Theorem A.8 . Put a short-lex ordering on word in X (as described in section A.2.1). Now subject the relator $x^a y^b x^c y^d$ to the Knuth-Bendix completion procedure (see Section A.2 or [KB70]). We look for critical pairs obtained by overlaps of $x^a y^b x^c y^d$ with itself and find that no self-overlaps will occur (and M will not be a group (see Theorem A.15)) unless both $d \leq \ b, a \leq \ c$. In the event that $a = c$ and $b = d$ the unique critical pair requires no resolution and $x^a y^b x^a y^b \to 1$ is a complete rewriting system for M and so y has no right inverse.

Suppose that $d \leq b, a \leq c$ and one of these inequalities is strict. Initialise a Knuth-Bendix procedure to discover that $x^a y^b x^c y^d y^{b-d} x^c y^d$ is both $y^{b-d} x^c y^d$ and $x^a y^b x^{c-a}$. Now $1 = x^a y^b y^b x^c y^d = x^a y^b x^a y^b x^{c-a} = y^{b-d} x^c y^d x^a y^d$. Thus either x or y has an inverse and so both do. □

Theorems A.9 and A.15 follow similar arguments using the Knuth-Bendix procedure to find if y has an inverse. In the case of Theorem A.15 no overlaps are possible and y will have no right inverse as all words equivalent to 1 begin with an x.

Proof of Theorem A.11. Observing that if there are less than three letters the result is clear as J. M. Howie has observed that Mon$< x, y \mid xy^2xy = 1 >$ is a group (see above). Suppose the letters are $x, y, a_1, a_2, \ldots, a_n$. Let w be the palindrome $a_1 a_2 \ldots a_n a_{n-1} a_{n-2} \ldots a_1$. Consider the non-palindromic relator $r = xwy^2x^2wy$ defining the monoid $M_n =$ Mon$< x, y, a_1, a_2, \ldots, a_n \mid r = 1 >$. Now in M_n we have

$$xwy^2x = xwy^2x \cdot 1 = xwy^2x \cdot xwy^2x^2wy = xwy^2x^2wy \cdot yx^2wy = 1 \cdot yx^2wy$$

so $1 = r = (yx^2wy)xwy = xwy(xwy^2x)$ and both x and y are invertible. Conjugation of r by suitable powers of x and y and their inverses yields that w has a two sided inverse. However w is a palindrome and so its letters themselves must be invertible and we are done. □

The rest of the proofs we omit making the following observations.

The idea of the proof of Theorem A.14 is to proceed as if we are performing a Todd-Coxeter enumeration [TC36, Joh90] in an attempt to find functions $x, y : \mathbb{N} \to \mathbb{N}$ which satisfy the relator. These functions which are defined inductively so that they are not both bijections, given the conditions of the theorem and so y will have no right inverse.

Proposition A.12 is proved by induction on $k + l + m + n$. The inductive step breaks into two cases: firstly when the pairs (k, m) and (l, n) are *comparable* (that is when either (a) $k \leq m$ and $l \leq n$ or (b) when $k \geq m$ and $l \geq n$) and secondly when k, l, m, n are incomparable (that is when they are not comparable). In the first case (for example (a)) we can replace the presentation for M by Mon$< x, y \mid x^k y^l = x^{m-k} y^{n-l} = 1 >$ without changing the value of the determinant and by induction we are finished. In the latter case when the pairs are incomparable, suppose without loss of generality that $k < m$ and $l > n$. Then we have $\det \begin{pmatrix} k & l \\ m & n \end{pmatrix} = kn - ml < 0$. Furthermore $x^{m-k} = x^m y^l = y^{l-n}$ whence $1 = x^m y^n = x^{m-k} y^k y^n = y^{l-n} x^k y^n$ and thus we have a right inverse to y. Thus as in the proof of Proposition A.7 we are done

The proof of Proposition A.13 is just as in the verification of the fact that the monoid Mon$< x, y \mid x^k y^l >$ is not a group (the condition ensures that $\eta\zeta \geq 0$ when $n \geq 0$ and any initial subword ζ of the image of r, whence r is mapped to $1 \in Z$).

A.4 More on Single Relator Monoids

Throughout this section we let $M = \text{Mon}< x, y \mid r = 1 >$. The following tells us some of the tale of whether M is a group or not.

Firstly two theorems from the paper by Smith [Smi97] that we saw in the previous section.

Theorem A.16. *Let M be as above and $r = x^a y^b x^c y^d$ where all the exponents are positive. The monoid M is a group if and only if $d \leq b$, $a \leq c$ and at least one of these inequalities is strict.*

Theorem A.17. *Let M be as above and $r = x^a y^b x^c y^d x^e y^f$ where all the exponents are positive. The monoid M is a group if and only if one of the following conditions holds:*

(a) $b = d, c = e, a \leq c, f \leq d$ and at least one of the inequalities is strict,

(b) $a \leq e, f \leq b$, and at least one of the inequalities is strict,

(c) $a = e, b = f, c \leq a, b \leq d$ and at least one of the inequalities is strict,

(d) $a = e, b = f, a \leq c, d \leq b$ and at least one of the inequalities is strict.

Now the natural extension to this,

Theorem A.18. *Let $r = x^a y^b x^c y^d x^e y^f x^g y^h$, with all the exponents positive. The monoid M is a group if and only if one of the following hold:*

(a) $a \leq g, h \leq b$ and at least one of the inequalities is strict,

(b) $a \leq e, b = f, g = c, h \leq d$ and at least one of the inequalities is strict,

(c) $a \leq c, b = d, c = e, d = f, g = e, h \leq f$ and at least one of the inequalities is strict,

(d) $a = e, b = f, c = g, d = h, a \leq c, d \leq b$ and at least one of the inequalities is strict,

(e) $a = g, b = h, c \leq a, b \leq d$ and at least one of the inequalities is strict,

(f) $a = g, b = h, a \leq e, b \leq f$ and at least one of the inequalities is strict,

(g) $a = g, b = h, a \leq e, f \leq b$ *and at least one of the inequalities is strict,*

(h) $a = g, b = h, c \leq e, f \leq d$ *and at least one of the inequalities is strict,*

(i) $a = g, b = h, a \leq e, b \leq d, b = f, a = c$ *and at least one of the inequalities is strict,*

(j) $a = g, b = h, c \leq e, b \leq f, d = f, a = e$ *and at least one of the inequalities is strict,*

(k) $a = g, b = h, c \leq a, b \leq f, d = b, c = e$ *and at least one of the inequalities is strict,*

(l) $a = g, b = h, a \leq c, f \leq d, b = d, c = e$ *and at least one of the inequalities is strict,*

(m) $a = g, b = h, a = c, b = d, a \leq e, f \leq b$ *and at least one of the inequalities is strict,*

(n) $a = g, b = h, a = c, b = d, a \leq e, b \leq f$ *and at least one of the inequalities is strict,*

(o) $a = g, b = h, a = c, b = d, e \leq a, b \leq f$ *and at least one of the inequalities is strict,*

(p) $a = g, b = h, a = e, b = f, a \leq c, d \leq b$ *and at least one of the inequalities is strict,*

(q) $a = g, b = h, a = e, b = f, a \leq c, b \leq d$ *and at least one of the inequalities is strict,*

(r) $a = g, b = h, a = e, b = f, c \leq a, b \leq d$ *and at least one of the inequalities is strict,*

(s) $a = g, b = h, c = e, f = d, a \leq c, b \leq d$ *and at least one of the inequalities is strict,*

(t) $a = g, b = h, c = e, f = d, c \leq a, b \leq d$ *and at least one of the inequalities is strict.*

While proving this theorem, we noticed some general results. Firstly recall that in [Smi97] it was mentioned that when there was a non-syllabic overlap in r, M was a group. We prove this here.

Lemma A.4.1. *Let M be as above and $r = \prod_{i=0}^{n} x^{a_i} y^{b_i}$ where a_i, b_i are all positive. Then M is a group if there is an non-syllabic overlap of r (recall that an overlap of words is syllabic if the overlapping segment consists of a collection of syllables of each word).*

The next lemma in some way generalises this:

Lemma A.4.2. *Let $\Sigma = \{x, y\}$, $W = \{r \in \Sigma^* | r = \prod_{i=0}^{n} x^{a_i} y^{b_i}\}$ and $|W| < \infty$. If there exists at least one non-syllabic overlap within W then we can find an inverse to x and y under the multiplication of Σ^*. Thus if M is a monoid and W is derived from r using a Knuth-Bendix procedure, M is a group.*

Now a 'negative' case, which generalises a result of Smith ([Smi97] or see A.15).

Proposition A.19. *Let X be a set of size at least two, and define the moniod M as $M = Mon< X \mid r^n = 1 >$. If no proper initial segment of r co-incides with a proper terminal segment of r then M is not a group.*

Corollary A.4.3. *Let $M = Mon< x, y \mid (x^a y^b)^n = 1 >$. Then M is not a group.*

Proofs

We prove the lemmas first to make the theorem easier.

Proof of lemma A.4.1. We begin by observing that if $r = \prod_{i=1}^{n} x^{a_i} y^{b_i}$ a non-syllabic overlap will occur when the following conditions hold:

$$\left.\begin{array}{ll} a_1 \leq a_j & b_1 = b_j, \\ a_i = a_{j+i} & b_i = b_{j+i}, 1 < i \leq n-j \\ a_{j-i} = a_j & b_j \leq b_{j-1} \end{array}\right\}$$

for some $1 < j \leq n-1$ and with one of the inequalities strict. Thus when we initiate a Knuth-Bendix procedure we get one overlap beginning at the jth position and this resolves to two equal words,

$$y^{b_{(j-1)}-b_n}(\prod_{k=j}^{n} x^{a_k} y^{b_k}) = (\prod_{l=1}^{n-j+1} x^{a_l} y^{b_l}) x^{a_{(n-j)}-a_1}$$

Now,

$$
\begin{aligned}
1 &= \prod_{i=1}^{n} x^{a_i} y^{b_i} \\
&= y^{b_{(j-1)}-b_n} (\prod_{k=j}^{n} x^{a_k} y^{b_k}) x_1^a y^{b_{n-j}} (\prod_{l=n-j+1}^{n} x^{a_l} y^{b_l}) \\
&= (\prod_{m=1}^{j-2} x^{a_m} y^{b_m}) x^{a_{j-1}} y^{b_n} (\prod_{p=1}^{n-j+1} x^{a_p} y^{b_p}) x^{a_{n-j}-a_1}
\end{aligned}
$$

So either x or y has a two sided inverse and so does the other. Thus M is a group

□

This gives us some monoid presentations which define groups.

Proof of Lemma A.4.2. Let $\alpha = U_1 x^a V y^b \in W$ and $\beta = x^c V y^d U_2 \in W$ where $U_i \in \Sigma^*$ are of the form $\prod_{j=1}^{n} x^{a_j} y^{b_j}$ and $V \in \Sigma^*$ is either empty or of the form $y^{b_0}(\prod_{j=1}^{n} x^{a_j} y^{b_j}) x^{a_{n+1}}$ with all the exponents positive, $c \leq a$ and $d \leq b$ and at least one of the inequalities is strict. Now initiate a Knuth-Bendix procedure on these words to find the overlap which will resolve into the rule:

$$U_1 x^{a-c} = y^{d-b} U_2$$

and so we get,

$$
\begin{aligned}
1 &= \alpha &= y^{d-b} U_2 x^c V y^b \\
&= \beta &= x^c V y^b U_1 x^{a-c}
\end{aligned}
$$

Thus either x or y will have a two sided inverse and then so does the other. □

Proof of Theorem A.18. As with the proof of Theorem A.16 and theorem A.17 we initiate a Knuth-Bendix procedure on the relator and see when this gives us inverses.

Lemma A.4.1 gives us conditions (a), (b), and (c) of non-syllabic overlaps of r.

Now we considering the syllabic overlaps analogous to these:

6 syllabic overlap: This gives us that $a = c, b = d, c = e, d = f, e = g, f = h$ *i.e.* $r = (x^a y^b)^4$ and this means that M is not a group.

4 syllabic overlap: This gives us that $a = e, b = f, c = g, d = h$ *i.e.* $r = (x^a y^b x^c y^d)^2 = 1$. This has the obvious overlap which resolves leaving us with a complete rewriting system. Thus y has no right inverse and M is not a group. Note in this case if $a \leq c, d \leq d$ and with at least one of these inequalities strict we would get a group (as this is a non-syllabic overlap) and this is condition (d).

Before looking at the 2 syllabic case, we introduce some (hopefully) useful notation. Let α and β be two words in M if these two words overlap with 4 syllables we will write this as $\alpha\beta(4)$ (not worrying whether this is syllabic or non-syllabic). Thus the two syllabic (or non-syllabic) case may be written $rr(2)$.

In the 2-syllabic case we have $a = g$ and $b = h$. The overlap implies that $x^a y^b x^c y^d x^e y^f = x^c y^d x^e y^f x^a y^b$, so we can write:

$$\begin{aligned} 1 &= x^a y^b x^c y^d x^e y^f x^a y^b \quad \text{(call this } \alpha) \\ &= x^c y^d x^e y^f (x^a y^b)^2 \quad \text{(call this } \beta) \\ &= (x^a y^b)^2 x^c y^d x^e y^f \quad \text{(call this } \gamma) \end{aligned}$$

We continue the Knuth-Bendix procedure by now looking at all the overlaps we have generated. These are (in the above notation):

$$\begin{array}{lll} \alpha\alpha(2), & \beta\alpha(2), & \gamma\alpha(6) \\ \alpha\beta(6), & \beta\gamma(2), & \gamma\beta(4) \\ \alpha\gamma(2), & \beta\gamma(4) & \end{array}$$

These all resolve yielding no further information. Thus M is not a group as y has no right inverse.

So we now consider the extra conditions on a, b, c, d, e and f that will force the words α, β, and γ to overlap. There are nine possible combinations of α, β, and γ and thus 27 ways in which to force an overlap (we have to consider each of 2, 4 and 6 syllables). These boil down to:

(i) $a = c = e, b = d = f$ [from $\alpha\alpha(6), \alpha\gamma(6), \beta\alpha(6), \gamma\beta(6)$]

(ii) $c \leq a, b \leq d$ [from $\alpha\beta(2), \beta\beta(2), \beta\alpha(4)$]

(iii) $a \leq e, b \leq f$ [from $\alpha\gamma(4)$]

(iv) $a \leq e, f \leq b$ [from $\gamma\alpha(2), \gamma\gamma(2)$]

(v) $c \leq e, f \leq d$ [from $\gamma\beta(2)$]

(vi) $a \leq e, b \leq d, b = f, a = c$ [from $\alpha\alpha(4), \beta\gamma(6)$]

(vii) $a \leq c, f \leq d, b = d, c = e$ [from $\gamma\alpha(4), \gamma\gamma(6)$]

(viii) $c \leq e, b \leq f, d = f, a = e$ [from $\alpha\beta(4), \beta\beta(6)$]

(ix) $c \leq a, b \leq f, b = d, a = e$ [from $\beta\beta(4), \gamma\gamma(4)$]

Forcing these to have one strict equality, we also have,

(x) $c = a, b = d$ [from (ii)]

(xi) $a = e, b = f$ [from (iii) and (iv)]

(xii) $c = e, d = f$ [from (v)]

Now by lemma A.4.2, we know that conditions (ii) to (ix) with strict inequalities will give us groups, and these are conditions (e) - (l).

Thus we only need to look at the conditions where there are syllabic overlaps, i.e. (i), (x), (xi) and (xii).

(i) is easy as it means that $(x^a y^b)^4 = 1$ and this we have already decided is not a group.

So let us consider (x), that is $c = a$ and $b = d$. We can now rewrite the relator as $1 = x^a y^b x^a y^b x^e y^f x^a y^b = x^a y^b x^e y^f (x^a y^b)^2 = (x^a y^b)^3 x^e y^f$, which we are calling α, β, and γ respectively. Resolving the overlaps we get (each word is underlined or over-lined):

Overlap	Rewrite	New rule
$\alpha\beta(2)$	$(x^a y^b)^2 x^e y^f x^a y^b x^e y^f (x^a y^b)^2$	$(x^a y^b)^2 x^e y^f = x^e y^f (x^a y^b)^2$
$\beta\beta(2)$	$x^a y^b x^e y^f x^a y^b x^a y^b x^e y^f (x^a y^b)^2$	$x^a y^b x^e y^f x^a y^b = x^e y^f (x^a y^b)^2$
$\beta\alpha(4)$	$x^a y^b x^e y^f (x^a y^b)^2 x^e y^f x^a y^b$	$x^a y^b x^e y^f = x^e y^f x^a y^b$

So now:

$$\begin{array}{rcll} 1 & = & (x^a y^b)^2 x^e y^f x^a y^b & \alpha \\ & = & x^a y^b x^e y^f (x^a y^b)^2 & \beta \\ & = & (x^a y^b)^3 x^e y^f & \gamma \\ & = & x^e y^f (x^a y^b)^3 & \delta \end{array}$$

Observing also that the following are equivalent:

$$\begin{aligned}
(x^a y^b)^2 x^e y^f &= x^e y^f (x^a y^b)^2 \\
x^a y^b x^e y^f x^a y^b &= x^e y^f (x^a y^b)^2 \\
x^a y^b x^e y^f &= x^e y^f x^a y^b
\end{aligned}$$

Now we look at the overlaps within α, β, γ and δ. These are

$$\begin{array}{llll}
\alpha\alpha(2) & \beta\alpha(2) & \gamma\alpha(6) & \delta\alpha(2) \\
\alpha\beta(6) & \beta\alpha(4) & \gamma\beta(4) & \delta\alpha(4) \\
\alpha\beta(6) & \beta\gamma(2) & \gamma\delta(2) & \delta\beta(2) \\
\alpha\gamma(2) & \beta\gamma(4) & & \delta\gamma(2) \\
\alpha\delta(4) & \beta\delta(6) & & \delta\gamma(4) \\
 & \beta\beta(2) & & \delta\gamma(6)
\end{array}$$

These all result in no new rules (we either have $z = z$ or get one of the derived rules). Thus M is not a group with these conditions. The only other possible conditions to force these to overlap are that:

1. $a \leq e$ and $f \leq b$
2. $a \leq e$ and $b \leq f$
3. $e \leq a$ and $b \leq f$
4. $a = c = e = g$ and $b = d = f = h$

(1), (2) and (3) with at least one strict inequality are groups by Lemma A.4.2 and correspond to (m), (n) and (o). We know (4) is not a group from Lemma A.19.

The case (xi) is similar but deals with the only the exponent letters being different, resulting in the extra conditions (p), (q) and (r) in the theorem.

The case of (xii) is slightly different: recall we now have $c = e, f = d$ as well as $a = g$ and $b = h$ so that α, β and γ become:

$$\begin{aligned} 1 &= x^a y^b (x^e y^f)^2 x^a y^b \quad (\alpha) \\ &= (x^e y^f)^2 (x^a y^b)^2 \quad (\beta) \\ &= (x^a y^b)^2 (x^e y^f)^2 \quad (\gamma) \end{aligned}$$

which then leads to $x^a y^b (x^c y^d)^2 = (x^c y^d)^2 x^a y^b$ and $(x^a y^b)^2 x^c y^d = x^c y^d (x^a y^b)^2$ resulting in the conditions $1 = \alpha = \beta = \gamma = x^c y^d (x^a y^b)^2 x^c y^d$ which is then complete. The only other conditions that will force overlaps are:

1. $a \leq c$ and $b \leq d$
2. $c \leq a$ and $b \leq d$
3. $a = c = e = g$ and $b = d = f = h$

Thus we only have groups in the case of (1) and (2): this is (s) and (t).

Thus we are done! □

A.4.1 To Terminate or not to Terminate

Having spent a little time using Derek Holt's kbmag programme (Knuth Bendix for Monoids and Automatic Groups) [Hol], I guessed the following.

Guess A.4.9. *Let M be as above, then a Knuth-Bendix procedure run on r will terminate if and only if M is a Monoid (i.e. if M is a group the procedure is infinite).*

After having spent some more time, this guess turns out to be partly unfounded. Looking at when M forms a group, we find that with a four relator this guess fails with a relator of $x^3 y^1 x^4 y^1$ (and in approximately 8% of all cases tried this is all four relators with indices between 1 and 4).

For the monoid case I have yet to find a counter example having searched through four and six relator monoids with indices ranging between 1 and 4. A more comprehensive search involving indices upto about two million and eight syllable relators is currently underway. However we still have:

Conjecture A.20. *Let $M = Mon< x, y \mid r = 1 >$ be a* **monoid** *where*

$$r = \prod_{i=1}^{n} x^{a_i} y^{b_i}.$$

A Knuth-Bendix procedure run on r will terminate in a finite time.

A possible idea for the proof is as follows. From our experience proving Theorems A.16, A.17 and A.18 we know that when we have a syllabic overlap a Knuth Bendix procedure resolving this seems to make $x^a y^b$ pairs commute (at worst). Thus the maximal rules we need to add are those which make such pairs commute. For a finite length word this is a finite number of extra rules.

A.5 A Universal Algorithm

We now attempt to describe and prove that there exists an algorithm to determine when $M = \text{Mon}< x, y \mid r = 1 >$, with $r = \prod_{i=1}^{n} x^{a_i} y^{b_i}$ and the a_i, b_i all positive, is a group.

A.5.1 Outline

Before writing down the procedure we attempt to outline the method. The general idea is to follow the methods of proof of Theorems A.16, A.17 and A.18. So we start by considering all conditions on the exponents giving non-syllabic overlaps of the word r. We then look at syllabic overlaps of decreasing length.

For each of these syllabic overlaps we check to see if they form complete rewriting systems using a Knuth-Bendix procedure. This of course may not terminate (but see conjecture A.20 above). If however it does terminate, we have a larger set of words with which to work, all equivalent to 1 in M. Within these words we now see how we can put extra conditions on the exponents to generate more overlaps.

Each of these new conditions we explore in turn (as above) until we will finally exhaust all possibilities.

We must of course answer the questions:

(i) Will it terminate?

(ii) Will the algorithm successfully cover all the cases?

Termination is not guaranteed because we are using a Knuth-Bendix procedure.

A.5.2 The Algorithm

```
MONOID_TEST(n)                ; n number of x^a y^b pairs
    if n=1 STOP
    For i=1 to n-1            ; deal with non-syllabic overlaps (Lemma A.4.1)
        record non-syllabic rule

    For i=1 to n-1
        MTS(n,i)              ; MTS considers syllabic overlaps of 2i syllables

MTS(n,i)                      ; n number of x^a y^b pairs, i size of overlap
                              ; sets up r explicitly
    r = ∏_{i=1}^n x^{a_i} y^{b_i} with (right conditions for given syllabic overlap)
    store r in 𝔑 and the conditions in 𝔍
                              ; 𝔍 has a list of equal indices
    GENERATE_RULES(𝔑, 𝔍)

GENERATE_RULES(𝔑, 𝔍)          ;𝔑 is collection of word(s), 𝔍 is set of equal indices
    if 𝔍⇒ a_1 = a_i, b_1 = b_i, ∀i STOP
                              ; we know that this isn't a group
    𝔑' = KNUTH_BENDIX(𝔑)      ; Run a Knuth-Bendix procedure on 𝔑,
                              ; not forcing overlaps,
                              ; (could record tested cases for efficiency)

    𝔖 = FORCE_OVERLAPS(𝔑'); Generate extra conditions on the exponents
                              ; so that we have considered all
                              ; possibilities of when two words in 𝔑 overlap

    SIMPLIFY(𝔖)               ; remove unless things, repeated conditions etc

    For each new non-syllabic condition in 𝔖
                              ; via Lemma A.4.2
        record condition
        generate syllabic condition
        remove non-syllabic condition

    SIMPLIFY(𝔖)
```

For each condition in $\mathfrak{S}$; recursive step

GENERATE_RULES($\mathfrak{R}$',$\mathfrak{I}$+*new condition*)

Theorem A.21. *If the algorithm* MONOID_TEST*(n) terminates then the monoid* M =*Mon*$< x, y \mid r = 1 >$ *with s*

$$r = \prod_{i=1}^{n} x^{a_i} y^{b_1}$$

and the a_i, b_i *all positive. Then* M *is a group if and only if* r *satisfies the conditions it generates.*

Proof. The algorithm starts by generating all the non-syllabic conditions and Lemma A.4.1 says that we need these.

Now for each syllabic overlap check the completeness of the rewriting system. This is step that may not terminate. However we are assuming that is will terminate (but see conjecture A.20 above). Thus we have a complete rewriting system and so M is not a group as y will have no right inverse, since all the words in $\mathfrak{R}$ (and hence all the words equivalent to 1) will begin with x.

We now look for extra conditions on these words to force more overlaps in $\mathfrak{R}$. We consider all possible overlaps of $\mathfrak{R}$ and record these conditions. By Lemma A.4.2 any of these overlaps which are non-syllabic will form a group, so we record these conditions. We also note the corresponding syllabic overlap of which we know nothing.

We then consider every new syllabic condition that we have just generated until we reach the case where the relator is $(x^{a_1} y^{b_1})^n$. This process will eventually stop, because each time a call is made to GENERATE_OVERLAPS a stricter condition is put on the words in $\mathfrak{R}$ (recall we are assuming that $\mathfrak{R}$ must be finite). This is as words in $\mathfrak{R}$ will have at most 2n syllables so that you will eventually have the condition that $a_1 = a_i, b_1 = b_i$ (or you will be able to generate more conditions on the exponents to repeat the process). Thus the recursion will terminate.

Thus MONOID_TEST will terminate.

We now must show we have considered all the possibilities. It is (hopefully) apparent that we must have as the only possible conditions for overlaps we have not looked at are those for 2n syllables (which are trivial) and when r doesn't overlap with itself (and this is not a group see Theorem A.14).

So we are done. □

We now only have to worry about what happens when the Knuth-Bendix procedure fails to complete. From the Knuth-Bendix theorem we know that even if this procedure fails to terminate we have a complete rewriting system, and so the monoid under consideration is not a group as all the words equivalent to 1 start with x. However this is not quite enough to get us where we want: the set $\mathfrak{R}$ in the algorithm is not as big as it should be because the Knuth-Bendix algorithm has not terminated. So from here on we do not necessarily exhaust the possibilities for overlaps. However the conjecture in the last section will give us termination and so we have some hope for this algorithm.

A.6 Further Thoughts

As we now have a way of deciding when a monoid with a single relator of the form $r = \prod_{i=1}^{n} x^{a_i} y^{b_i}$ is a group we now consider how this may be extended. Observe that we have Proposition A.19 which tells us a negative case.

First a positive result:

Proposition A.22. *Let $M = Mon < a_1, a_2, \ldots, a_n \mid w = 1 >$ where w is of the form $a_1^{b_1} a_2^{b_2} \ldots a_n^{b_n} u a_n^{c_n} a_{n-1}^{c_{n-1}} a_{n-2}^{c_{n-2}} \ldots a_1^{c_1}$ with $u \in Mon< a_1, a_2, \ldots, a_n \mid \_ >$, and all exponents positive. Thus w is not necessarily a palindrome. Then M is a group.*

Of course we needn't stop here, we could take this to its limit:

Proposition A.23. *Let $M = Mon < a_1, a_2, \ldots, a_n \mid w = 1 >$. Let u_i and v_i be words in $Mon< a_1, a_2, \ldots, a_n \mid \_ >$ only containing a_j such that $j \leq i$. Let w is of the form $a_1^{b_1} a_2^{b_2} u_2 a_3^{b_3} \ldots u_{n-1} a_n^{b_n} u_n a_n^{c_n} v_{n-1} a_{n-1}^{c_{n-1}} v_{n-2} a_{n-2}^{c_{n-2}} \ldots v_2 a_2^{c_2} a_1^{c_1}$. Thus w is not necessarily a palindrome. Then M is a group.*

Proposition A.24. *Let $M = Mon < x, y, a_1, a_2, \ldots, a_n \mid r = 1 >$ and let w be as in the proposition above. If $r = \prod_{i=1}^{n} x^{a_i} w^b y^{c_i}$ then M is a group if $N = Mon< x, y \mid \prod_{i=1}^{n} x^{a_i} y^{c_i} >$ is a group with the inequalities in the relevant conditions strict.*

In the above, the b (which are the exponents of w) need not all be the same, however they must ensure that we get the necessary overlaps, so it is simpler to keep things like this.

Now let $M = \mathrm{Mon}< a_1, a_2, \ldots, a_n \mid r = 1 >$. If r does not contain all the a_i then M is not a group (trivially). Let $\pi_{ij} : M \to N_1$ be defined by:

$$\left.\begin{array}{rcl} (a_k)\pi_{ij} & = & 1 \text{ if } i \neq k \neq j \\ (a_k)\pi_{ij} & = & x \text{ if } i = k \\ (a_k)\pi_{ij} & = & 1 \text{ if } k = j \\ \text{if } U = u_1 u_2 \ldots u_m, (U)\pi_{ij} & = & (u_1)\pi_{ij}(u_2)\pi_{ij} \ldots (u_m)\pi_{ij} \end{array}\right\}$$

where $N_1 = \mathrm{Mon}< x, y \mid (r)\pi_{ij} = 1 >$. Thus π_{ij} is a homomorphism (observe that $(1_M)\pi_{ij} = (r)\pi_{ij} = 1_N$ and we are done).

We also define $\varpi_i : M \to N_2$ as follows:

$$\left.\begin{array}{rcl} (a_k)\varpi_i & = & x \text{ if } k \leq i \\ (a_k)\varpi_i & = & y \text{ if } k > i \\ \text{if } U = u_1u_2\ldots u_m, (U)\varpi_i & = & (u_1)\varpi_i(u_2)\varpi_i\ldots(u_m)\varpi_i \end{array}\right\}$$

where $N_2 = \text{Mon}< x, y \mid (r)\varpi_i = 1 >$. Thus ϖ_i is a homomorphism (observe that $(1_M)\varpi_i = (r)\varpi_i = 1_N$ and we are done).

Now with these two homomorphisms defined we have:

Proposition A.25. *If $M = Mon< a_1, a_2, \ldots, a_n \mid r = 1 >$ then if N_1 or N_2 (as defined above) are not groups then neither is M.*

This can be quite useful. For example if $M = \text{Mon}< a_1, a_2, \ldots, a_n \mid r = 1 >$ and $r = \prod_{i=1}^n a_i^{b_i}$ where all the b_i are positive then M is not a group as $M = \text{Mon}< x, y \mid x^k y^l = 1 >$ is not a group. Thus by Proposition A.25 we have that M as above with $r = \prod_{j=1}^m (\prod_{i=1}^n a_i^{b_i})$ is not a group. Thus we have just proved,

Proposition A.26. *Let $M = Mon< a_1, a_2, \ldots, a_n \mid r = 1 >$ and*

$$r = \prod_{j=1}^{n}(\prod_{i=1}^{n} a_i^{b_i}).$$

Then M is not a group.

Now consider the basic example $M = \text{Mon}< x, y, z \mid x^a y^b z^c x^d y^e z^f >$ which by Proposition A.24 is a group when $a < d$, $b = e$ and $f < c$. Using the above homomorphism ϖ_i we find to be a group the following conditions must hold (the conditions are just Theorem A.16).

$$\begin{array}{lll} a \leq d & e + f \leq b + c & \text{(one of these inequalities must be strict)} \\ a + b \leq d + e & f \leq c & \text{(one of these inequalities must be strict)} \end{array}$$

So as $a \leq d$ we have $a + b \leq d + b \leq d + e$ so that $b \leq e$. Also as $f \leq c, e + f \leq e + c \leq b + c$ so $e \leq b$. However now we have both $b \leq e$ and $e \leq b$, so $b = e$. Thus for M to be a group $M = \text{Mon}< x, y, z \mid x^a y^b z^c x^d y^b z^f >$ and have $a \leq d$, $f \leq c$ with one of these inequalities strict. In fact now applying the π_{ij}'s we see that in fact we must have $a < d, f < c$ otherwise we will not have groups under these homomorphisms. Note that we have not proved here that a M of this form is a group, only that to be a group M must be of this form, however in this case Proposition A.24 will make this a group.

You can of course play this game with $\mathrm{Mon}< w, x, y, z \mid w^a x^b y^c z^d w^e x^f y^g z^h >$ and find under ϖ_i that you need,

$a \leq e$	$f + g + h \leq b + c + d$	(one of these inequalities must be strict)
$a + b \leq e + f$	$g + h \leq c + d$	(one of these inequalities must be strict)
$a + b + c \leq e + f + g$	$h \leq d$	(one of these inequalities must be strict)

so that you need $a \leq e, b \leq e, c \leq g$ and $h \leq d, g \leq c, f \leq d$. So that $c = g$, now applying π_{yz} we force $h < d$, applying π_{xy} we force $b < f$ and applying π_{wy} we force $a < e$. Thus M can only be group when $a < e, b < f, c = g, h < d$. In fact these conditions will not allow overlaps (need b=f). Thus starting a Knuth-Bendix procedure on $w^a x^b y^c z^d w^e x^f y^g z^h$ will find no overlaps. Thus y will not have a right inverse (as all words equivalent to 1 start with an x) and so M will not be a group.

We could now ask how this continues. A sufficiently diligent reader may wish to explore this further.

Proofs

We will not prove Proposition A.22 as it is a consequence of Proposition A.23.

Proof of proposition A.23. Observing firstly that if we can find a right inverse and a left inverse for an element we can find a two sided inverse (if $xy = 1$ and $zx = 1$ then set $w = zxy$ and we have $wx = zxy \cdot x = zx \cdot yx = 1 = xy = x \cdot zx \cdot y = x \cdot zxy = xw$). We have that a_1 will have a two sided inverse (as it both starts and ends the word). Now conjugate w by appropriate powers of a_1 to find that a_2 has a 2-sided inverse (or more specifically a right and a left inverse as c_2 does not have to be equal to b_2). Now carry on this process observing that very time you encounter a u_i or v_i in the way it consists of elements you have inverses for and so you may conjugate it away. Thus all the a_i will have a two sided inverse, and so M is a group. □

Now we prove,

Proof of proposition A.24. If we have strict inequalities in the relator

$$r = \prod_{i=1}^{n} x^{a_i} y^{c_i}$$

of $M = \text{Mon}< x, y \mid r >$ we get an inverse to x and y without the need for conjugation. So we may obtain in the same way an two sided inverse for x and y using $\prod_{i=1}^{n} x^{a_i} w^b y^{c_i}$ as the w^b will not affect any overlapping we require. So by conjugation by appropriate powers of x and y we can get a two sided inverse for w (or a left and a right inverse, and by the remark above a two sided inverse). Thus by peeling the layers off w we may obtain an inverse for all the a_i □

We note the brief proof of:

Proof of proposition A.25. If M was a group the N_i would be also. Thus if N_i is not a group neither is M. □